Chiral Differential Operators: Formal Loop Group Actions and Associated Modules

Pokman Cheung

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Abstract

Chiral differential operators (CDOs) are closely related to the geometry of loop spaces and the quantum theory of 2-dimensional σ-models. This paper investigates two topics about CDOs on smooth manifolds. In the first part, we study how a Lie group action on a smooth manifold can be lifted to a "formal loop group action" on an algebra of CDOs; this turns out to be a condition on the equivariant first Pontrjagin class. The case of a principal bundle receives particular attention and gives rise to a vertex algebra that plays a fundamental role in the theory. In the second part, we introduce a construction of modules over CDOs using the said "formal loop group actions" and semi-infinite cohomology. Intuitively, these modules should have a geometric meaning in terms of the hypothetical "formal loop spaces". The first example we study leads to a new and more conceptual construction of an arbitrary algebra of CDOs. The other example, called the spinor module, may be useful for formalizing the physical meaning of the Witten genus for smooth string manifolds.

§1. Introduction

Algebras of chiral differential operators, or *algebras of CDOs*, are certain vertex algebras closely related to the geometry of loop spaces and the quantum theory of 2-dimensional σ-models. Locally, the algebra of CDOs on e.g. an affine space $\mathbb{A}^d$ is an elementary conformal vertex algebra (a $\beta\gamma$-system) and it has some appropriate holomorphic and smooth versions (see §2.1–§2.2). The global construction of CDOs, first given by Gorbounov, Malikov and Schechtman, shares various features with string geometry and σ-models. In particular, the obstructions for a complex manifold M to admit a sheaf of holomorphic CDOs $\mathcal{D}^{\mathrm{ch}}_M$ are certain refinements of $c_1(M)$ and $c_2(M)$; and if M is closed, then the genus-1 partition function of the conformal vertex superalgebra $H^*(M;\mathcal{D}^{\mathrm{ch}}_M)$ is essentially the Witten genus of M. [GMS00, Che11] In the realm of algebraic geometry, Kapranov and Vasserot have provided an interpretation of CDOs in terms of the notion of *formal loop spaces* they introduced. [KV04, KV06] Intuitively, the "formal loop space of X" is the formal neighborhood of the subspace of constant loops (i.e. X itself) inside the space of all loops, but this has yet to be made precise for, say, smooth manifolds. Meanwhile, physicists have explained how holomorphic CDOs describe the so-called large volume limit of the half-twisted σ-model. [Kap06, Wit07, Tan06] More recently, using his own mathematical formulation of quantum field theory, Costello has given a new construction of the Witten genus for complex manifolds, referring to it as an "analytic avatar" of holomorphic CDOs. [Cos10, Cos11]

This paper investigates two related topics about CDOs on *smooth* manifolds: (i) their interactions with smooth Lie group actions and (ii) construction of geometrically meaningful modules. First, anticipating an interpretation of CDOs in terms of "formal loops" (à la Kapranov-Vasserot), we study how a Lie group action on a manifold can be lifted to a "formal loop group action" on an algebra of CDOs. This turns out to be a condition on the equivariant first Pontrjagin class. The case of a principal bundle receives particular attention and gives rise to a vertex algebra that plays an important role in the rest of the paper (perhaps in the overall theory as well). Using the abovementioned "formal loop group actions" and semi-infinite cohomology, we then introduce a construction of modules over CDOs. Intuitively, each of these modules should be the space of sections of some vector bundle over a "formal loop space", as a module over "differential operators". The first example we study leads to a new and more conceptual construction of an arbitrary algebra of CDOs. The other example, called the spinor module, may be useful for formalizing the physical meaning of the Witten genus for smooth string manifolds.

The following is a more detailed overview of the paper.

§ 1.1. Overview of the paper. §2 recalls the definition of an algebra of CDOs on a smooth manifold (Definition 2.3) as well as the general construction and classification of these vertex algebras (Theorems 2.5 and 2.6). The explicit generators-and-relations construction described here should be compared to the more conceptual one in §6.

Let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra and λ an invariant symmetric bilinear form on $\mathfrak{g}$. Notice that λ determines a centrally extended loop algebra $\hat{\mathfrak{g}}_\lambda$ (§3.1) and also a vertex algebra $V_\lambda(\mathfrak{g})$ (Example A.10). If $\mathfrak{g}$ is simple, we will use the more common notation $V_k(\mathfrak{g})$ in place of $V_{k\lambda_0}(\mathfrak{g})$, where $k \in \mathbb{C}$ and λ_0 is the normalized Killing form. Let us introduce the following terminology: a *formal loop group action* of level λ on a vector space W is a $\hat{\mathfrak{g}}_\lambda$-action on W whose restriction to $\mathfrak{g} \subset \hat{\mathfrak{g}}_\lambda$ integrates into a G-action; and if W is a vertex algebra then we also require that $\mathfrak{g}$ acts by inner derivations. (In the main text this is called an *inner* $(\hat{\mathfrak{g}}_\lambda, G)$*-action*; see Definition 3.4.) Notice that in the latter case the action is induced by a map of vertex algebras $V_\lambda(\mathfrak{g}) \to W$.

Suppose P is a smooth manifold with a smooth G-action and $\mathcal{D}^{\mathrm{ch}}(P)$ is an algebra of CDOs on P. Conjecturally, there should be a description of $\mathcal{D}^{\mathrm{ch}}(P)$ in terms of the "formal loop space of P". This motivates the main result in §3:

Theorem 3.11. *The G-action on P lifts to a formal loop group action on $\mathcal{D}^{\mathrm{ch}}(P)$ of level λ if and only if*

$$8\pi^2 p_1(P)_G = \lambda(P)$$

where $p_1(P)_G$ is the equivariant first Pontrjagin class and $\lambda(P)$ is the image of λ under the characteristic map $(\mathrm{Sym}^2\mathfrak{g}^\vee)^G \cong H^4(BG) \to H^4_G(M)$. Moreover, the said action is primary with respect to a suitably

chosen conformal vector of $\mathcal{D}^{\mathrm{ch}}(P)$.

The key ideas behind this result are: the use of the Cartan model for equivariant de Rham cohomology (recalled in §3.2); and the observation that the G-action on P and the vertex algebra structure of $\mathcal{D}^{\mathrm{ch}}(P)$ together determine a Cartan cocycle for $p_1(P)_G$ (Lemma 3.6). In fact, there is a more refined statement detailing a bijection between the formal loop group actions in question and certain Cartan cochains.

From now, P is the total space of a principal G-bundle $\pi : P \to M$ and the algebra of CDOs $\mathcal{D}^{\mathrm{ch}}(P)$ is equipped with a formal loop group action $V_\lambda(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}(P)$. §4 is a more detailed study of the vertex algebra $\mathcal{D}^{\mathrm{ch}}(P)$ in this situation. First we consider the centralizer subalgebra

$$\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}} = C\big(\mathcal{D}^{\mathrm{ch}}(P), V_\lambda(\mathfrak{g})\big)$$

i.e. the subalgebra invariant under the formal loop group action. While the structure of $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ is not fully understood, a notable part of it is equivalent to the Atiyah algebroid $(C^\infty(M), \mathcal{T}(P)^G)$ (§4.4). For this reason the vertex algebra $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ has more interesting modules compared to an algebra of CDOs on M. This will become more apparent later.

In the example $P = G$ with G simple, the two G-actions on itself by left and right multiplications can be lifted to two *commuting* formal loop group actions $V_k(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}(G)$ and $V_{-k-2h^\vee}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}(G)$, where $k \in \mathbb{C}$ is arbitrary and $h^\vee$ is the dual Coxeter number (§3.12). Hence there is an inclusion of, say, $V_k(\mathfrak{g})$ into $C\big(\mathcal{D}^{\mathrm{ch}}(G), V_{-k-2h^\vee}(\mathfrak{g})\big)$. If $k \neq -h^\vee$, then $V_k(\mathfrak{g})$ has a Sugawara vector and it turns out to be also conformal for the above centralizer (§4.5). This seems to suggest that the said inclusion may in fact be surjective, but the author has yet to find either a proof or a counterexample.

In the case where π is a principal frame bundle of TM, we find a better description of $\mathcal{D}^{\mathrm{ch}}(P)$ such that (among other things) the formal loop group action becomes manifest. This occupies the second half of §4 and leads to the following result, which is central to the rest of the paper:

Theorem 4.14. *Suppose we have a principal G-bundle $\pi : P \to M$, a representation $\rho : G \to SO(\mathbb{R}^d)$ and an isomorphism $P \times_\rho \mathbb{R}^d \cong TM$; a connection Θ on π that induces the Levi-Civita connection on TM; an invariant symmetric bilinear form λ on $\mathfrak{g}$; and a basic 3-form H on P with $dH = (\lambda + \lambda_{\mathsf{ad}} + \lambda_\rho)(\Omega \wedge \Omega)$, where Ω is the curvature of Θ (see also §1.2). These data determine a conformal vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$, which is generated by primary fields of weights 0 and 1, and equipped with a primary formal loop group action of level λ. For its detailed definition see the main text.*

Even though this vertex algebra arises as a particular algebra of CDOs, it should really be regarded as a more fundamental object for both conceptual and aesthetic reasons. In fact, from vertex algebras of the form $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$, there is a natural way to recover *all* algebras of CDOs and also construct many other interesting objects (see below). Moreover, the definition of $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ is arguably more appealing than that of an arbitrary algebra of CDOs.

§5 introduces the following construction: given a vector space W with a formal loop group action of level $-\lambda - \lambda_{\mathsf{ad}}$, we can define a module over $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ as follows (Definition 5.5 & Lemma 5.6)

$$\Gamma^{\mathrm{ch}}(\pi, W) := H^{\frac{\infty}{2}+0}\big(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, \mathcal{D}^{\mathrm{ch}}(P) \otimes W\big). \tag{1.1}$$

The notation $H^{\frac{\infty}{2}+*}$ refers to *semi-infinite (a.k.a. BRST) cohomology*, which can be defined using the Feigin complex (recalled in §5.2–§5.3). This is analogous to the construction of associated vector bundles of $\pi : P \to M$. In fact, $\Gamma^{\mathrm{ch}}(\pi, W)$ should be the space of sections of a vector bundle over the "formal loops of M", as a module over "differential operators". Notice that if W is a vertex algebra, then so is $\Gamma^{\mathrm{ch}}(\pi, W)$ and there is a map of vertex algebras $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}} \to \Gamma^{\mathrm{ch}}(\pi, W)$.

In our first example of (1.1), π is a principal frame bundle of TM and $W = \mathbb{C}$ (hence $\lambda = -\lambda_{\mathsf{ad}}$). By Theorem 4.14, the definition of $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ depends on the choice of some $H \in \pi^*\Omega^3(M)$ that satisfies $dH = \operatorname{Tr} \rho(\Omega) \wedge \rho(\Omega)$. The main result in §6 is a new description of CDOs as alluded to earlier:

Theorem 6.11. *The vertex algebra $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C}) = H^{\frac{\infty}{2}+0}(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P))$ is an algebra of CDOs on M. Up to isomorphism every algebra of CDOs on M is of this form.*

The proof consists of two parts: first we identify the two lowest weights of $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ (§6.2 & §6.5) and work out their structure (Proposition 6.7); then we use a certain property of its conformal vector to deduce that

$\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ is completely determined by its two lowest weights (Proposition 6.9 & Corollary 6.10). There is also an extension of Theorem 6.11 for supermanifolds, including a special case that recovers the chiral de Rham complex (§6.12).

In our second and last example of (1.1), G is a spin group $\mathrm{Spin}_{2d'}$, π is a spin structure on M and $W = S$ is the spinor representation of $\widehat{\mathfrak{so}}_{2d'}$. It follows from Theorem 4.14 that the definition of the *spinor module* $\Gamma^{\mathrm{ch}}(\pi, S)$ depends on the choice of some $H \in \pi^*\Omega^3(M)$ that satisfies $dH = \frac{1}{2}\mathrm{Tr}\,(\Omega \wedge \Omega)$ (§7.3). Geometrically, $\Gamma^{\mathrm{ch}}(\pi, S)$ should be the space of sections of the "spinor bundle on the formal loops of M". In §7 we carry out an analysis of $\Gamma^{\mathrm{ch}}(\pi, S)$ that parallels the one in §6 and culminates in a more explicit description of it in terms of generating data (i.e. a subspace and three types of fields) and relations. This is summarized in Theorem 7.14. Notice that our work in §7 should be viewed mainly as preparation for further study. The author *speculates* that the spinor module $\Gamma^{\mathrm{ch}}(\pi, S)$ always admits an action of the $(0, 1)$ superconformal algebra and thereby provides at least a partial mathematical account of the physical interpretation of the Witten genus (§7.16).

The appendix reviews the notion of vertex algebroids and their relations with vertex algebras. Although vertex algebroids have a rather complicated definition, they serve as a convenient tool for dealing with the vertex algebras in this paper.

§ 1.2. Notations and conventions. In this paper, every vertex algebra V is graded by nonnegative integers which we call *weights*; its component of weight k is denoted by V_k and its weight operator by L_0, i.e. $L_0|_{V_k} = k$. For any $u \in V$, we always write the Fourier modes of its vertex operator as u_k, $k \in \mathbb{Z}$, such that u_k has weight $-k$. For any conformal vector $\nu \in V$ that we consider, $\nu_0 = L_0$.

Given a smooth manifold M, we write $C^\infty(M)$, $\mathcal{T}(M)$, $\Omega^n(M)$ for its spaces of smooth $\mathbb{C}$-valued functions, vector fields and n-forms. Also, all ordinary cohomology groups and their equivariant versions have complex coefficients. Given a Lie algebra $\mathfrak{g}$ and a finite-dimensional representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, we write λ_ρ for the invariant symmetric bilinear form on $\mathfrak{g}$ given by $\lambda_\rho(A, B) = \mathrm{Tr}\,\rho(A)\rho(B)$. In particular, λ_{ad} is the Killing form. Square brackets [ ] are used for supercommutators between operators of any parities, while curly brackets { } are reserved for a different use (see §A.3). Repeated indices are always implicitly summed over all possible values, unless a specific range is indicated.

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§2. CDOs on Smooth Manifolds

A sheaf of CDOs is a sheaf of vertex algebras locally modelled on a very basic vertex algebra (known as the $\beta\gamma$-system), and provides a mathematical approximation to a quantum field theory of much interest. [MSV99, GMS00, GMS04, KV06, Kap06, Wit07, Tan06] This section reviews the construction and classification of sheaves of CDOs on a smooth manifold, following the formulation in [Che11].

§ 2.1. The algebra of CDOs on $\mathbb{A}^d$. Let d be a positive integer. Define a unital associative algebra $\mathcal{U}$ with the following generators and relations

$$b^i_n,\ a_{i,n},\ n\in\mathbb{Z},\ i=1,\ldots,d, \qquad [a_{i,n},b^j_m]=\delta^j_i\delta_{n,-m}, \qquad [b^i_n,b^j_m]=0=[a_{i,n},a_{j,m}]\,. \tag{2.2}$$

(This is an example of a Weyl algebra.) The commutative subalgebra $\mathcal{U}_+$ generated by $\{b^i_n\}_{n>0}$ and $\{a_{i,n}\}_{n\geq 0}$ has a trivial representation $\mathbb{C}$. The induced $\mathcal{U}$-module

$$\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d) := \mathcal{U}\otimes_{\mathcal{U}_+}\mathbb{C}$$

has the structure of a vertex algebra. The vacuum is $\mathbf{1}=1\otimes 1$. The infinitesimal translation operator T and weight operator L_0 are determined by

$$\begin{array}{lll} T\mathbf{1}=0, & [T,b^i_n]=(1-n)b^i_{n-1}, & [T,a_{i,n}]=-na_{i,n-1} \\ L_0\mathbf{1}=0, & [L_0,b^i_n]=-nb^i_n, & [L_0,a_{i,n}]=-na_{i,n} \end{array}$$

The fields (or vertex operators) of $b^i_0\mathbf{1}\in\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_0$ and $a_{i,-1}\mathbf{1}\in\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_1$ are respectively

$$\textstyle\sum_n b^i_n z^{-n}, \qquad \sum_n a_{i,n}z^{-n-1}$$

which determine the fields of other elements by the Reconstruction Theorem [FB04]. This vertex algebra has a family of conformal elements of central charge $2d$, namely [1]

$$a_{i,-1}b^i_{-1}\mathbf{1}+T^2f, \qquad f\in\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_0=\mathbb{C}[b^1_0,\ldots,b^d_0]\cdot\mathbf{1}.$$

The vertex algebra $\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)$ is freely generated by its associated vertex algebroid (see §A.5 and §A.7). To describe the latter, consider the affine space $\mathbb{A}^d=\operatorname{Spec}\mathbb{C}[b^1,\cdots,b^d]$ and identify the functions, 1-forms and vector fields on $\mathbb{A}^d$ with the following subquotients of $\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)$:

- $\mathcal{O}(\mathbb{A}^d)=\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_0$ via $b^i=b^i_0\mathbf{1}$, $b^ib^j=b^i_0b^j_0\mathbf{1}$, etc.
- $\Omega^1(\mathbb{A}^d)\subset\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_1$ via $db^i=b^i_{-1}\mathbf{1}$
- $\mathcal{T}(\mathbb{A}^d)=\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_1/\Omega^1(\mathbb{A}^d)$ via $\partial_i=\partial/\partial b^i=$ coset of $a_{i,-1}\mathbf{1}$

Under these identifications, the vertex algebroid associated to $\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)$ is of the form

$$\big(\mathcal{O}(\mathbb{A}^d),\Omega^1(\mathbb{A}^d),\mathcal{T}(\mathbb{A}^d),\bullet,\{\ \},\{\ \}_\Omega\big).$$

The extended Lie algebroid structure consists of the usual differential on functions, Lie bracket on vector fields, Lie derivations by vector fields on functions and 1-forms, and pairing between 1-forms and vector fields. Using the splitting

$$s:\mathcal{T}(\mathbb{A}^d)\to\mathcal{D}^{\mathrm{ch}}(\mathbb{A}^d)_1, \qquad X=X^i\partial_i\ \mapsto\ a_{i,-1}X^i \tag{2.3}$$

the rest of the vertex algebroid structure, according to (A.2), reads as follows

$$X\bullet f=(\partial_jX^i)(\partial_if)db^j, \quad \{X,Y\}=-(\partial_jX^i)(\partial_iY^j), \quad \{X,Y\}_\Omega=-(\partial_k\partial_jX^i)(\partial_iY^j)db^k \tag{2.4}$$

These expressions do not seem to have any obvious global meaning; however, see Theorem 2.5.

[1] This vertex algebra is the tensor product of d copies of the $\beta\gamma$-system. It admits other conformal elements that define different conformal weights. [Kac98]

§ 2.2. The sheaf of CDOs on $\mathbb{R}^d$. Now regard $b^1, \dots, b^d$ as the coordinates of $\mathbb{R}^d$. The ($\mathbb{C}$-valued) smooth functions, 1-forms and vector fields on an open set $W \subset \mathbb{R}^d$ form an extended Lie algebroid just as in §2.1, and the expressions in (2.4) again define a vertex algebroid

$$\big(C^\infty(W), \Omega^1(W), \mathcal{T}(W), \bullet, \{\ \}, \{\ \}_\Omega\big).$$

The vertex algebra it freely generates (see §A.7) will be denoted by $\mathcal{D}^{\mathrm{ch}}(W)$. This vertex algebra also has a family of conformal elements of central charge $2d$, namely

$$\partial_{i,-1} db^i + \frac{1}{2} T\omega, \qquad \omega \in \Omega^1(W),\ d\omega = 0. \tag{2.5}$$

When W varies, we obtain a sheaf of conformal vertex algebras $\mathcal{D}^{\mathrm{ch}}$ on $\mathbb{R}^d$.

Definition 2.3. A **sheaf of CDOs** on a smooth manifold M of dimension d is a sheaf of vertex algebras $\mathcal{V}$ with the following properties:

- its weight-zero component is $\mathcal{V}_0 = C^\infty_M$, and
- each point of M has a neighborhood U such that $(U, \mathcal{V}|_U)$ is isomorphic to $(W, \mathcal{D}^{\mathrm{ch}}|_W)$ for some open set $W \subset \mathbb{R}^d$.

A **conformal structure** on $\mathcal{V}$ is an element $\nu \in \mathcal{V}(M)$ such that, under each isomorphism as postulated above, $\nu|_U \in \mathcal{V}(U)$ corresponds to one of the conformal elements (2.5) of $\mathcal{D}^{\mathrm{ch}}(W)$.

In order to state the results on the construction and classification of sheaves of CDOs, let us introduce a notation that will also appear often in the sequel.

Definition 2.4. Let M be a smooth manifold. Given a connection ∇ on TM and $X \in \mathcal{T}(M)$, define an operator $\nabla^t X \in \Gamma(\operatorname{End} TM)$ by

$$(\nabla^t X)(Y) := \nabla_X Y - [X, Y], \qquad Y \in \mathcal{T}(M).$$

Notice that if ∇ is torsion-free, then $\nabla^t X = \nabla X$.

Theorem 2.5. *[Che11] Let M be a smooth manifold of dimension d.*

(a) Given a connection ∇ on TM with curvature R and $H \in \Omega^3(M)$ satisfying $dH = \operatorname{Tr}(R \wedge R)$, there is a sheaf of vertex algebroids $\big(C^\infty_M, \Omega^1_M, \mathcal{T}_M, \bullet, \{\ \}, \{\ \}_\Omega\big)$ on M defined by the following expressions

$$\begin{aligned} X \bullet f \quad &:= (\nabla X) f \\ \{X, Y\} \quad &:= -\operatorname{Tr}(\nabla^t X \cdot \nabla^t Y) \\ \{X, Y\}_\Omega &:= \operatorname{Tr}\Big(-\nabla(\nabla^t X) \cdot \nabla^t Y + \nabla^t X \cdot \iota_Y R - \iota_X R \cdot \nabla^t Y \Big) + \frac{1}{2} \iota_X \iota_Y H \end{aligned} \tag{2.6}$$

and the sheaf of vertex algebras it freely generates (see §A.7) is a sheaf of CDOs on M, which we denote by $\mathcal{D}^{\mathrm{ch}}_{M,\nabla,H}$. Up to isomorphism, every sheaf of CDOs on M is of this form.

(b) There is a one-to-one correspondence between conformal structures on $\mathcal{D}^{\mathrm{ch}}_{M,\nabla,H}$ and $\omega \in \Omega^1(M)$ satisfying $d\omega = \operatorname{Tr} R$. Given such ω, the corresponding conformal structure, which we denote by ν^ω, has the local expression

$$\nu^\omega|_U = \sigma_{i,-1}\sigma^i + \frac{1}{2} \operatorname{Tr}\big(\Gamma^\sigma_{-1}\Gamma^\sigma - \Gamma^\sigma_{-2}\mathbf{1}\big) + \sigma^i([\sigma_j, \sigma_k])\sigma^k_{-1}(\Gamma^\sigma)^j{}_i + \frac{1}{2}\omega_{-2}\mathbf{1} \tag{2.7}$$

where $U \subset M$ is an open subset, $\sigma = (\sigma_1, \dots, \sigma_d)$ any $C^\infty(U)$-basis of $\mathcal{T}(U)$, $(\sigma^1, \dots, \sigma^d)$ the dual basis of $\Omega^1(U)$, and $\Gamma^\sigma \in \Omega^1(U) \otimes \mathfrak{gl}_d$ the connection 1-form of ∇ with respect to σ (i.e. $\nabla\sigma_i = (\Gamma^\sigma)^j{}_i \otimes \sigma_j$). This conformal structure has central charge $2d = 2\dim M$ and the property that

$$\nu^\omega_1 \alpha = 0 \ \text{ for } \alpha \in \Omega^1(M), \qquad \nu^\omega_1 X = \operatorname{Tr} \nabla^t X - \omega(X) \ \text{ for } X \in \mathcal{T}(M). \quad \square \tag{2.8}$$

Remarks. (i) By Theorem 2.5, a smooth manifold M admits sheaves of CDOs if and only if $p_1(M)$ is trivial in de Rham cohomology, while conformal structures always exist. For example, if ∇ is orthogonal with respect to a Riemannian metric, then $\operatorname{Tr} R = 0$ and a conformal structure can be defined using, say, $\omega = 0$. However, this result generalizes to cs-manifolds (supermanifolds with $\mathbb{C}$-valued structure sheaf) in which case the obstruction to conformal structures may well be nontrivial. [Che11]

(ii) In [Che11], we only obtained a local expression of ν^ω in terms of local coordinate vector fields, but that implies the more general expression (2.7) by a straightforward calculation.

(iii) In the original work [GMS04] as well as in [Che11], sheaves of CDOs and conformal structures were constructed by gluing local data. In the smooth case, this culminates in a description by generators and relations (or generating fields and OPEs) as seen above. However, the expressions in (2.6) and (2.7) do not seem very inspiring. Later in §6, we will obtain a more conceptual description of CDOs using semi-infinite cohomology.

Theorem 2.6. *[Che11] Let $\mathcal{D}^{\mathrm{ch}}_{M,\nabla,H}$ and $\mathcal{D}^{\mathrm{ch}}_{M,\nabla,H'}$ be sheaves of CDOs on a smooth manifold M constructed as in Theorem 2.5a; denote by $\bullet$, $\{\ \}$, $\{\ \}_\Omega$ (resp. $\{\ \}'_\Omega$) the structure maps determined by ∇ and H (resp. H') as in (2.6).*

(a) There is a one-to-one correspondence between isomorphisms $\mathcal{D}^{\mathrm{ch}}_{M,\nabla,H} \xrightarrow{\sim} \mathcal{D}^{\mathrm{ch}}_{M,\nabla,H'}$ that restricts to the identity on C^∞_M and $\beta \in \Omega^2(M)$ satisfying $d\beta = H' - H$. Given such β, the corresponding isomorphism is induced by an isomorphism of sheaves of vertex algebroids (see §A.8)

$$(\mathrm{id}, \Delta_\beta) : \big(C^\infty_M, \Omega^1_M, \mathcal{T}_M, \bullet, \{\ \}, \{\ \}_\Omega\big) \to \big(C^\infty_M, \Omega^1_M, \mathcal{T}_M, \bullet, \{\ \}, \{\ \}'_\Omega\big)$$

where $\Delta_\beta : \mathcal{T}_M \to \Omega^1_M$ is given by $\Delta_\beta(X) = \frac{1}{2}\iota_X\beta$.

(b) Every isomorphism described above respects the correspondence in Theorem 2.5b. □

Remarks. (i) By Theorems 2.5a and 2.6a, if M admits sheaves of CDOs, their isomorphism classes form an $H^3(M)$-torsor. (ii) Since each sheaf of CDOs $\mathcal{D}^{\mathrm{ch}}_{M,\nabla,H}$ is fine, for most purposes it suffices to (and we will) work only with the vertex algebra of global sections $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$, which will be called an **algebra of CDOs** on M. The reader should keep in mind that by construction

$$\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)_0 = C^\infty(M), \qquad \mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)_1 = \Omega^1(M) \oplus \mathcal{T}(M). \tag{2.9}$$

For a description of the higher weights, see §A.11.

Lemma 2.7. *[Che11] Consider an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$ on a smooth manifold M. For any $\alpha \in \Omega^1(M)$ and $X \in \mathcal{T}(M)$, we have $\alpha_0 X = -\iota_X d\alpha$. Moreover, $\alpha_0 = 0$ on $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$ if and only if $d\alpha = 0$.* □

§3. Formal Loop Group Actions on CDOs

This section investigates the condition under which a Lie group action on a manifold lifts to a projective "formal loop group action" on an algebra of CDOs. This turns out to be a condition on the equivariant first Pontrjagin class of the manifold. The said action provides evidence of a conjectural interpretation of (smooth) CDOs in terms of "formal loops", in the vein of [KV06].

§ 3.1. Setting: manifold with a Lie group action. Throughout this section, let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and λ an invariant symmetric bilinear form on $\mathfrak{g}$. Recall the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: writing $A \otimes t^n$ as A_n, the Lie bracket is given by $[A_n, B_m] = [A, B]_{n+m}$. Also recall that λ determines a central extension $\hat{\mathfrak{g}}_\lambda = L\mathfrak{g} \oplus \mathbb{C}$ with

$$[A_n, B_m] = [A, B]_{n+m} + n\lambda(A, B)\delta_{n+m,0}, \quad A, B \in \mathfrak{g}, \ n, m \in \mathbb{Z}.$$

Let P be a smooth manifold with a smooth right G-action. Later we will specialize to the case of a principal bundle, but at the moment P can be any right G-manifold. The left G-action on $C^\infty(P)$ determines and is determined by a map of Lie algebras $\mathfrak{g} \to \mathcal{T}(P)$. The vector field generated by $A \in \mathfrak{g}$ will be written as $A^P \in \mathcal{T}(P)$.

§ 3.2. The equivariant de Rham complex. Recall that $H^*_G(P) = H^*(EG \times_G P)$ can be computed by the Cartan model $(\Omega^*_G(P), d_G)$. [GS99] The graded algebra of Cartan cochains is given by

$$\Omega^*_G(P) = \bigoplus_{k \geq 0} \Omega^k_G(P), \qquad \Omega^k_G(P) = \bigoplus_{2i+j=k} \big(\mathrm{Sym}^i \mathfrak{g}^\vee \otimes \Omega^j(P)\big)^G.$$

Let us adopt a convention: regard each $\xi \in \Omega^*_G(P)$ as a G-equivariant polynomial map $\xi : \mathfrak{g} \to \Omega^*(P)$ and write its value at $A \in \mathfrak{g}$ as ξ_A. The Cartan differential then reads

$$(d_G\xi)_A = d\xi_A - \iota_{A^P}\xi_A.$$

The characteristic map $H^*(BG) \to H^*_G(P)$ is represented by the inclusion $(\mathrm{Sym}^* \mathfrak{g}^\vee)^G \hookrightarrow \Omega^*_G(P)$. Given $\eta \in (\mathrm{Sym}^* \mathfrak{g}^\vee)^G = H^*(BG)$, its image will be denoted by $\eta(P) \in H^*_G(P)$.

§ 3.3. CDOs with a G-action. Choose a G-invariant connection ∇ on TP, with curvature tensor R. This means for $A \in \mathfrak{g}$ we have $L_{A^P}\nabla = 0$, or equivalently

$$\nabla(\nabla^t A^P) = -\iota_{A^P} R \tag{3.1}$$

in terms of Definition 2.4. [2] Assume $p_1(P) = 0$ and choose $H \in \Omega^3(P)^G$ such that $dH = \mathrm{Tr}\,(R \wedge R)$. By Theorem 2.5a, ∇ and H determine an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(P)$, which is freely generated by a vertex algebroid (see also §A.7)

$$\big(C^\infty(P), \Omega^1(P), \mathcal{T}(P), \bullet, \{\ \}, \{\ \}_\Omega\big);$$

when there is no risk of confusion, we simply write $\mathcal{D}^{\mathrm{ch}}(P)$. Clearly, the G-invariance of ∇ and H implies G-equivariance of the structure maps $\bullet$, $\{\ \}$, $\{\ \}_\Omega$, so that the G-action on $C^\infty(P) = \mathcal{D}^{\mathrm{ch}}(P)_0$ extends to a G-action on $\mathcal{D}^{\mathrm{ch}}(P)$.

Without loss of generality, assume that $\mathrm{Tr}\, R = 0$. (For example, this is true if ∇ is orthogonal with respect to a Riemannian metric.) Choose $\omega \in \Omega^1(P)^G$ such that $d\omega = 0$. By Theorem 2.5b, ω determines a G-invariant conformal structure ν^ω on $\mathcal{D}^{\mathrm{ch}}(P)$. The Fourier modes of the associated Virasoro field will be denoted by L^ω_n, $n \in \mathbb{Z}$. Soon we will make a more specific choice of ω.

[2] For any $X \in \mathcal{T}(P)$ we have $L_X\nabla = \nabla(\nabla^t X) + \iota_X R$. This is proved as follows: for $Y, Z \in \mathcal{T}(P)$

$$\begin{aligned}
(L_X\nabla)_Y Z &= [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z] \\
&= \nabla_Y \nabla_X Z - \nabla_Y [X, Z] - \nabla_X \nabla_Y Z + [X, \nabla_Y Z] + \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&= \nabla_Y (\nabla^t X)(Z) - (\nabla^t X)(\nabla_Y Z) + R_{X,Y} Z = \big(\nabla_Y(\nabla^t X) + R_{X,Y}\big)(Z)
\end{aligned}$$

Remark. The G-invariance of ∇ and H, as well as that of other geometric data to appear later, can always be achieved by averaging over G with respect to the Haar measure.

Definition 3.4. Given a vertex algebra V, by an **inner $\hat{\mathfrak{g}}_\lambda$-action** on V we simply mean a map of vertex algebras from $V_\lambda(\mathfrak{g})$ to V, and it is an **inner $(\hat{\mathfrak{g}}_\lambda, G)$-action** if its induced $\mathfrak{g}$-action on V (see below) integrates into a G-action.

Remarks. (i) By definition of $V_\lambda(\mathfrak{g})$ (see Example A.10), any map $V_\lambda(\mathfrak{g}) \to V$ is determined by its component of weight one, i.e. a linear map $\mathfrak{g} = V_\lambda(\mathfrak{g})_1 \to V_1$. Taking the zeroth modes then yields a map of Lie algebras from $\mathfrak{g}$ to the inner derivations of V. This is the induced $\mathfrak{g}$-action referred to above. (ii) Now we may state the goal of this section more precisely: find the condition under which the given G-action on $\mathcal{D}^{\mathrm{ch}}(P)_0 = C^\infty(P)$ extends to an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}(P)$.

§ 3.5. Cartan cochains associated to CDOs. The G-action on P and the vertex algebroid structure associated to $\mathcal{D}^{\mathrm{ch}}(P)$ together determine two Cartan cochains of degree 4, namely

$$\begin{aligned} &\chi^{2,2} \in \big(\mathfrak{g}^\vee \otimes \Omega^2(P)\big)^G, && \chi^{2,2}_A := \{A^P, -\}_\Omega = \mathrm{Tr}\,(\nabla^t A^P \cdot R) - \tfrac{1}{2}\iota_{A^P} H \\ &\chi^{4,0} \in \big(\mathrm{Sym}^2\mathfrak{g}^\vee \otimes C^\infty(P)\big)^G, && \chi^{4,0}_A := \{A^P, A^P\} = -\mathrm{Tr}\,(\nabla^t A^P \cdot \nabla^t A^P) \end{aligned} \tag{3.2}$$

Indeed, by (2.6) and (3.1), the operator $\{A^P, -\}_\Omega : \mathcal{T}(P) \to \Omega^1(P)$ may be viewed as the indicated 2-form, and the G-invariance of ∇, H implies the G-equivariance of $\chi^{2,2}$, $\chi^{4,0}$. Moreover, the G-action on P and the conformal structure ν^ω on $\mathcal{D}^{\mathrm{ch}}(P)$ together determine a Cartan cochain of degree 2, namely

$$\chi^{2,0} \in \big(\mathfrak{g}^\vee \otimes C^\infty(P)\big)^G, \qquad \chi^{2,0}_A := L^\omega_1 A^P = \mathrm{Tr}\,\nabla^t A^P - \omega(A^P) \tag{3.3}$$

in view of (2.8). Indeed, the G-invariance of ∇, ω implies the G-equivariance of $\chi^{2,0}$.

Lemma 3.6. *(a) The Cartan cochain $2\chi^{2,2} + \chi^{4,0}$ is closed and represents $8\pi^2 p_1(P)_G \in H^4_G(P)$. (b) The Cartan cochain $\chi^{2,0}$ is exact. In fact, it is trivial with a suitable choice of conformal structure ν^ω.*

Proof. (a) According to [BGV92, BT01], the Cartan cochain $A \mapsto \mathrm{Tr}\,(R - \nabla^t A^P)^2$ is closed and represents $-8\pi^2 p_1(P)_G$. Our claim then follows from the calculation

$$\mathrm{Tr}\,(R - \nabla^t A^P)^2 = dH - 2\,\mathrm{Tr}\,(\nabla^t A^P \cdot R) + \mathrm{Tr}\,(\nabla^t A^P \cdot \nabla^t A^P) = (d_G H)_A - 2\chi^{2,2}_A - \chi^{4,0}_A$$

where we have used $dH = \mathrm{Tr}\,(R \wedge R)$ and (3.2).

(b) According to [BGV92, BT01] again, the Cartan cochain $A \mapsto \mathrm{Tr}\,(R - \nabla^t A^P) = -\mathrm{Tr}\,\nabla^t A^P$ is exact. Since we have $(d_G\omega)_A = d\omega - \omega(A^P) = -\omega(A^P)$, this proves the exactness of (3.3). This also means there exists some $\omega' \in \Omega^1(P)^G$ such that

$$-\mathrm{Tr}\,\nabla^t A^P = (d_G\omega')_A = d\omega' - \omega'(A^P)$$

which is equivalent to two equations: firstly $d\omega' = 0$, so that ω' also determines a conformal structure $\nu^{\omega'}$; and secondly $L^{\omega'}_1 A^P = 0$. □

Remarks. (i) The closedness of $2\chi^{2,2} + \chi^{4,0}$ is equivalent to a pair of equations

$$d\chi^{2,2}_A = 0, \qquad d\chi^{4,0}_A = 2\iota_{A^P}\chi^{2,2}_A, \qquad \text{for } A \in \mathfrak{g} \tag{3.4}$$

which can also be verified directly from our assumptions on ∇ and H. (ii) From now on we assume that the conformal structure ν^ω has been chosen such that $L^\omega_1 A^P = 0$ for $A \in \mathfrak{g}$, and write the associated Virasoro operators more simply as L_n.

Preparation. The following sequence of results all concern lifting the given map of Lie algebras $\mathfrak{g} \to \mathcal{T}(P)$ to a linear map of the form

$$\mathfrak{g} \to \mathcal{D}^{\mathrm{ch}}(P)_1 = \mathcal{T}(P) \oplus \Omega^1(P), \qquad A \mapsto A^P + h^{2,1}_A \tag{3.5}$$

where $h^{2,1} : \mathfrak{g} \to \Omega^1(P)$ is assumed to be G-equivariant. In other words, $h^{2,1}$ is a Cartan cochain.

Proposition 3.7. *The linear map (3.5) induces a map of Lie algebras as follows*

$$\mathfrak{g} \to \operatorname{Der}\mathcal{D}^{\mathrm{ch}}(P), \qquad A \mapsto \big(A^P + h^{2,1}_A\big)_0 \tag{3.6}$$

if and only if the closed 2*-form* $\chi^{2,2}_A - dh^{2,1}_A$ *is* G*-invariant for each* $A \in \mathfrak{g}$*.*

Proof. For $A, B \in \mathfrak{g}$, we have the following calculation

$$\begin{aligned}\big[\big(A^P + h^{2,1}_A\big)_0, \big(B^P + h^{2,1}_B\big)_0\big] &= \Big([A^P, B^P] + \{A^P, B^P\}_\Omega + L_{A^P}h^{2,1}_B - L_{B^P}h^{2,1}_A\Big)_0 \\ &= \Big([A,B]^P + h^{2,1}_{[A,B]}\Big)_0 + \Big(\iota_{B^P}\chi^{2,2}_A - L_{B^P}h^{2,1}_A\Big)_0\end{aligned}$$

using (A.3), (3.2) and the G-equivariance of $h^{2,1}$. By Lemma 2.7, the second term vanishes if and only if

$$0 = d\big(\iota_{B^P}\chi^{2,2}_A - L_{B^P}h^{2,1}_A\big) = L_{B^P}\big(\chi^{2,2}_A - dh^{2,1}_A\big)$$

where we have used (3.4). This proves our claim. $\square$

The map of Lie algebras $\mathfrak{g} \to \mathcal{T}(P)$ extends in an obvious way to a map of extended Lie algebroids $i : (\mathbb{C}, 0, \mathfrak{g}) \to (C^\infty(P), \Omega^1(P), \mathcal{T}(P))$. Now we would like to extend it further to a map of vertex algebroids and hence a map of vertex algebras (see Example A.10 and §A.8).

Proposition 3.8. *The linear map (3.5) determines a map of vertex algebroids as follows*

$$(i, h^{2,1}) : (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0) \to \big(C^\infty(P), \Omega^1(P), \mathcal{T}(P), \bullet, \{\ \}, \{\ \}_\Omega\big)$$

and hence a map of vertex algebras $V_\lambda(\mathfrak{g}) \to \mathcal{D}^{\mathrm{ch}}(P)$ *if and only if, for each* $A \in \mathfrak{g}$*, the closed* 2*-form* $\chi^{2,2}_A - dh^{2,1}_A$ *is* G*-horizontal and the following equation holds:*

$$\chi^{4,0}_A + 2h^{2,1}_A(A^P) = \lambda(A, A). \tag{3.7}$$

Proof. Let $A, B \in \mathfrak{g}$. According to Definition A.4, the linear map $h^{2,1} : \mathfrak{g} \to \Omega^1(P)$ defines a map between the said vertex algebroids if and only if

$$\begin{aligned}\{A^P, B^P\} \ &= \lambda(A,B) - h^{2,1}_A(B^P) - h^{2,1}_B(A^P) \\ \{A^P, B^P\}_\Omega &= -L_{A^P}h^{2,1}_B + L_{B^P}h^{2,1}_A - dh^{2,1}_A(B^P) + h^{2,1}_{[A,B]}\end{aligned}$$

By (3.2) the first equation is equivalent to (3.7). By (3.2) again and the G-equivariance of $h^{2,1}$, the second equation can be rewritten as $\iota_{B^P}(\chi^{2,2}_A - dh^{2,1}_A) = 0$. This proves our claim. In fact, the G-horizontality of $\chi^{2,2}_A - dh^{2,1}_A$ always implies that the left side of (3.7) is locally constant, since

$$d\big(\chi^{4,0}_A + 2\iota_{A^P}h^{2,1}_A\big) = 2\iota_{A^P}\chi^{2,2}_A - 2\iota_{A^P}dh^{2,1}_A + 2L_{A^P}h^{2,1}_A = 2\iota_{A^P}\big(\chi^{2,2}_A - dh^{2,1}_A\big)$$

by virtue of (3.4) and the G-equivariance of $h^{2,1}$. Therefore if P is connected, the horizontal condition guarantees that (3.7) must hold for some $\lambda \in (\mathrm{Sym}^2\mathfrak{g}^\vee)^G$. $\square$

Proposition 3.9. *The map of Lie algebras (3.6) integrates into a homomorphism* $G \to \operatorname{Aut}\mathcal{D}^{\mathrm{ch}}(P)$ *if and only if the closed* 2*-form* $\chi^{2,2}_A - dh^{2,1}_A$ *vanishes for each* $A \in \mathfrak{g}$*.*

Proof. First we compute the one-parameter subgroup of automorphisms generated by the inner derivation $(A^P + h^{2,1}_A)_0$. Since $\mathcal{D}^{\mathrm{ch}}(P)$ is freely generated by a vertex algebroid, it suffices to compute the automorphisms at weights 0 and 1. For the following computations, keep (A.3) in mind. For $f \in C^\infty(P)$ and $\alpha \in \Omega^1(P)$ we simply have

$$\big(A^P + h^{2,1}_A\big)^n_0 f = (A^P)^n f \qquad \Rightarrow \qquad e^{t(A^P + h^{2,1}_A)_0} f = e^{tA} \cdot f \tag{3.8}$$

$$\big(A^P + h^{2,1}_A\big)^n_0 \alpha = L^n_{A^P}\alpha \qquad \Rightarrow \qquad e^{t(A^P + h^{2,1}_A)_0} \alpha = e^{tA} \cdot \alpha \tag{3.9}$$

where $\cdot$ refers to the given G-actions. For $X \in \mathcal{T}(P)$ we first have

$$\big(A^P + h^{2,1}_A\big)_0 X = [A^P, X] + \{A^P, X\}_\Omega - \iota_X dh^{2,1}_A = [A^P, X] + \iota_X(\chi^{2,2}_A - dh^{2,1}_A)$$

by Lemma 2.7 and (3.2). Then it follows by induction and the G-equivariance of $\chi^{2,2}$, $h^{2,1}$ that

$$\begin{aligned} & \big(A^P + h^{2,1}_A\big)_0^n X = L^n_{A^P} X + n L^{n-1}_{A^P} \iota_X(\chi^{2,2}_A - dh^{2,1}_A), \qquad n \geq 1 \\ \Rightarrow \qquad & e^{t(A^P + h^{2,1}_A)_0} X = e^{tA} \cdot X + t e^{tA} \cdot \iota_X(\chi^{2,2}_A - dh^{2,1}_A) \end{aligned} \tag{3.10}$$

This finishes the computation of the automorphisms.

If $\chi^{2,2}_A - dh^{2,1}_A$ vanishes for all $A \in \mathfrak{g}$, it follows from (3.8)–(3.10) that (3.6) integrates into the G-action described in §3.3. Conversely, assume that (3.6) integrates into a G-action. By (3.10), $\chi^{2,2}_A - dh^{2,1}_A$ must vanish whenever $e^A = 1$. Since every element of G lies in a torus, the subset $\{A \in \mathfrak{g} \,|\, e^A = 1\}$ spans $\mathfrak{g}$, so that $\chi^{2,2}_A - dh^{2,1}_A$ must in fact vanish for all $A \in \mathfrak{g}$. ☐

Remark. Notice that according to the above proof, whenever (3.6) is integrable, the resulting G-action on $\mathcal{D}^{\mathrm{ch}}(P)$ can only be the one described in §3.3.

§ 3.10. Comparing the three conditions. Recall the conditions encountered respectively in Propositions 3.7, 3.8 and 3.9:

(i) $\chi^{2,2}_A - dh^{2,1}_A$ is G-invariant for $A \in \mathfrak{g}$
(ii) $\chi^{2,2}_A - dh^{2,1}_A$ is G-horizontal for $A \in \mathfrak{g}$
(iii) $\chi^{2,2}_A - dh^{2,1}_A = 0$ for $A \in \mathfrak{g}$

In general, (iii) ⇒ (ii) ⇒ (i); the second implication follows from (3.4). From another point of view, a map of vertex algebras as in Proposition 3.8 always induces a map of Lie algebras as in Proposition 3.7, but the other implication may seem somewhat surprising. In the case $\mathfrak{g}$ is semisimple, so that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, all three conditions are equivalent.

Here is the main result of this section.

Theorem 3.11. *The G-action on P lifts to an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}(P)$ if and only if*

$$8\pi^2 p_1(P)_G = \lambda(P).$$

Moreover, this action is primary with respect to the chosen conformal structure ν^ω.

Proof. Recall Definition 3.4 and Example A.10. First of all, an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action as described is determined by a linear map $\mathfrak{g} \to \mathcal{D}^{\mathrm{ch}}(P)_1$ of the form $A \mapsto A^P + h^{2,1}_A$, where $h^{2,1} \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$. By Propositions 3.8 and 3.9, the precise conditions on $h^{2,1}$ are

$$\left\{ \begin{array}{l} \chi^{2,2}_A = dh^{2,1}_A \\ \chi^{4,0}_A = \lambda(A, A) - 2h^{2,1}_A(A^P) \end{array} \right. \qquad \Longleftrightarrow \qquad 2\chi^{2,2} + \chi^{4,0} = \lambda + 2d_G h^{2,1}. \tag{3.11}$$

By Lemma 3.6a, this proves the first claim. The second claim is simply a restatement of Lemma 3.6b and a subsequent remark there. ☐

Remark. In the sequel, $h^{2,1}$ will be called the **associated Cartan cochain** of the inner $(\hat{\mathfrak{g}}_\lambda, G)$-action.

Example 3.12. CDOs on a Lie group. Let G be a compact simple Lie group. In this discussion, A, B always mean general elements of $\mathfrak{g} = T_e G$. The following notations will be used:

- A^ℓ (resp. A^r) is the left-invariant (resp. right-invariant) vector field on G extending A;
- θ is the Maurer-Cartan form on G, i.e. $\theta \in \Omega^1(G) \otimes \mathfrak{g}$ such that $\theta(A^\ell) = A$;
- λ_{ad} is the Killing form and $\lambda_0 = (2h^\vee)^{-1}\lambda_{\mathsf{ad}}$ the normalized Killing form on $\mathfrak{g}$, where $h^\vee$ is the dual Coxeter number.

There is a canonical isomorphism $H^4(BG) \cong H^3(G)$, where both spaces are one-dimensional. The isomorphism is represented by

$$\lambda \in (\mathrm{Sym}^2\mathfrak{g}^\vee)^G \quad \longmapsto \quad -\frac{1}{6}\,\lambda(\theta \wedge [\theta \wedge \theta]) \in \Omega^3(G)$$

and we will use λ_0, or the corresponding 3-form H_0, as the respective basis.

Now we construct a conformal algebra of CDOs on G. Let ∇ be the flat connection on TG such that $\nabla A^\ell = 0$. Given $s \in \mathbb{C}$, there is a closed 3-form sH_0. By Theorems 2.5a and 2.6a, ∇ and sH_0 together define an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}_s(G) = \mathcal{D}^{\mathrm{ch}}_{\nabla,sH_0}(G)$, and every algebra of CDOs on G is up to isomorphism of this form. Notice that the vertex algebroid structure of $\mathcal{D}^{\mathrm{ch}}_s(G)$ includes such special cases as

$$\{A^\ell, A^\ell\} = -2h^\vee\lambda_0(A,A), \qquad \{A^\ell, -\}_\Omega = \frac{s}{4}\,\lambda_0(A,[\theta\wedge\theta]) \tag{3.12}$$

$$\{A^r, -\} = 0, \qquad \{A^r, -\}_\Omega = \frac{s}{4}\,\lambda_0(\theta(A^r),[\theta\wedge\theta]) \tag{3.13}$$

By Theorem 2.5b, the trivial 1-form defines a conformal structure ν of central charge $2\dim\mathfrak{g}$. If $J_1, J_2, \cdots$ are a basis of $\mathfrak{g}$ and $\theta^1, \theta^2, \cdots$ the respective components of θ, we can write $\nu = J^\ell_{a,-1}\theta^a$. Also, ν has the property that $L_1A^\ell = L_1A^r = 0$.

First consider the action of G on itself by right multiplication. The induced map of Lie algebras $\mathfrak{g} \to \mathcal{T}(G)$ sends A to A^ℓ. Both ∇ and H_0 are invariant under this action. Define $h^\ell \in (\mathfrak{g}^\vee \otimes \Omega^1(G))^G$ by $h^\ell_A = -\frac{s}{2}\lambda_0(A,\theta)$. In view of (3.12), h^ℓ satisfies (3.11) with $\lambda = (-s-2h^\vee)\lambda_0$. Therefore h^ℓ is the Cartan cochain associated to an inner $(\hat{\mathfrak{g}}, G)$-action

$$V_{-s-2h^\vee}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_s(G)\,; \tag{3.14}$$

denote its image by $V_{-s-2h^\vee}(\mathfrak{g})^\ell$. This action is primary with respect to ν.

Now consider also the action of G on itself by inverse left multiplication, i.e. each $g \in G$ acts via left multiplication by g^{-1}. The induced map of Lie algebras $\mathfrak{g} \to \mathcal{T}(G)$ sends A to $-A^r$. Since left- and right-invariant vector fields commute, ∇ and H_0 are invariant under this action as well. Define $h^r \in (\mathfrak{g}^\vee \otimes \Omega^1(G))^G$ by $h^r_A = \frac{s}{2}\lambda_0(\theta(A^r),\theta)$. In view of (3.13), h^r satisfies (3.11) with $\lambda = s\lambda_0$. [3] Therefore h^r is the Cartan cochain associated to an inner $(\hat{\mathfrak{g}}, G)$-action

$$V_s(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_s(G)\,; \tag{3.15}$$

denote its image by $V_s(\mathfrak{g})^r$. This action is also primary with respect to ν.

Remark. For both G-actions, since $H^*_G(G) = H^*(\mathrm{pt})$, the condition in Theorem 3.11 is trivial for any $\lambda \in (\mathrm{Sym}^2\mathfrak{g}^\vee)^G$, so that in fact there are many other inner $(\hat{\mathfrak{g}}, G)$-actions on $\mathcal{D}^{\mathrm{ch}}_s(G)$.

[3] Notice that $(dh^r_A)(B^r, C^r) = \frac{s}{2}\lambda_0(A,[B,C]) = \chi^{2,2}_A(B^r,C^r)$ for $B, C \in \mathfrak{g}$, so that indeed $dh^r_A = \chi^{2,2}_A$.

§4. CDOs on Principal Bundles

In this section, the object of interest is an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}(P)$ on the total space of a smooth principal bundle $P \to M$, equipped with a projective "formal loop group action" in the sense of §3. The first goal is to understand the subalgebra of $\mathcal{D}^{\mathrm{ch}}(P)$ invariant under the said action. In subsequent sections we will construct and study modules over this invariant subalgebra. The second (and more technical) goal is to give an alternative description of $\mathcal{D}^{\mathrm{ch}}(P)$ in the case $P \to M$ is a principal frame bundle. This vertex algebra will play a central role in the rest of the paper: it is not merely a particular algebra of CDOs, but should be regarded as a more fundamental object from which all algebras of CDOs (as well as other related constructions) derive.

§ 4.1. Setting: principal bundle. Let G be again a compact connected Lie group and $\pi : P \to M$ a smooth principal G-bundle. Identify $H^*_G(P)$ with $H^*(M)$ via π^*. Choose a connection Θ on π, i.e. some $\Theta \in (\Omega^1(P) \otimes \mathfrak{g})^G$ such that $\Theta(A^P) = A$ for $A \in \mathfrak{g}$; then its curvature is $\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$. Keep in mind that Θ is equivalent to a G-equivariant vector bundle decomposition $TP = T_hP \oplus T_vP$, with $T_hP = \ker \Theta$ and $T_vP = \ker \pi_*$; and Ω measures the non-integrability of the subbundle T_hP. Given $X \in \mathcal{T}(M)$, denote its horizontal lift by $\widetilde{X} \in \mathcal{T}_h(P)^G$. Let us extend the notation $(\cdot)^P$ (see §3.1) $C^\infty(P)$-linearly to represent the isomorphism $C^\infty(P) \otimes \mathfrak{g} \cong \mathcal{T}_v(P)$. For $X, Y \in \mathcal{T}(M)$ notice that

$$[\widetilde{X}, \widetilde{Y}] - \widetilde{[X, Y]} = -\Omega(\widetilde{X}, \widetilde{Y})^P. \tag{4.1}$$

Also recall the notations introduced in §3.2.

Consider an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}(P) = \mathcal{D}^{\mathrm{ch}}_{\nabla,H}(P)$ defined as in §3.3. Let $\lambda \in (\mathrm{Sym}^2\mathfrak{g}^\vee)^G$. Recall from Definition 3.4 and the proof of Theorem 3.11 the meaning of an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}(P)$ and its associated Cartan cochain. It will be understood without further comment that any inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}(P)$ considered below extends the given G-action on $\mathcal{D}^{\mathrm{ch}}(P)_0 = C^\infty(P)$.

Preparation. Given a representation $\rho : G \to GL(V)$, let us denote the induced map of Lie algebras also by ρ and define an invariant symmetric bilinear form on $\mathfrak{g}$ by $\lambda_\rho(A, B) = \mathrm{Tr}\,\rho(A)\rho(B)$. In particular, λ_{ad} is the Killing form. If we regard V as a G-equivariant vector bundle over a point and identify $H^4_G(\mathrm{pt})$ with $(\mathrm{Sym}^2\mathfrak{g}^\vee)^G$, then $-8\pi^2 p_1(V)_G = \lambda_\rho$.

Lemma 4.2. *$8\pi^2 p_1(P)_G = 8\pi^2 p_1(M) - \lambda_{\mathsf{ad}}(P)$.*

Proof. Consider the G-equivariant decomposition $TP = T_hP \oplus T_vP$. On one hand, since $T_hP \cong \pi^*TM$ and π^* identifies $H^*_G(P)$ with $H^*(M)$, we have $p_1(T_hP)_G = p_1(TM)$. On the other hand, $T_vP \cong P \times \mathfrak{g}$ with G acting on $\mathfrak{g}$ in the adjoint representation, so that $-8\pi^2 p_1(T_vP) = \lambda_{\mathsf{ad}}(P)$. This proves the lemma. □

Hence Theorem 3.11 specializes to our current setting as follows.

Corollary 4.3. *The G-action on P lifts to an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}(P)$ if and only if*

$$8\pi^2 p_1(M) = (\lambda + \lambda_{\mathsf{ad}})(P). \quad \square \tag{4.2}$$

Remark. For convenience, we will refer to $\mathcal{D}^{\mathrm{ch}}(P)$ as a **principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebra** when it is equipped with an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action, or a **principal $(\hat{\mathfrak{g}}, G)$-algebra** if we do not wish to specify λ.

§ 4.4. The invariant subalgebra. Suppose $\mathcal{D}^{\mathrm{ch}}(P)$ is a principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebra, i.e. there is an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action $V_\lambda(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}(P)$ defined by some associated Cartan cochain $h \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$. Consider the centralizer subalgebra [Kac98, FB04]

$$\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}} := C\big(\mathcal{D}^{\mathrm{ch}}(P), V_\lambda(\mathfrak{g})\big) \subset \mathcal{D}^{\mathrm{ch}}(P).$$

This is the subalgebra whose fields are $\hat{\mathfrak{g}}_\lambda$-invariant. The weight-zero component consists of $f \in C^\infty(P)$ such that $(A^P + h_A)_0 f = A^P f = 0$ for $A \in \mathfrak{g}$, namely

$$\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}_0 = C^\infty(P)^G = \pi^* C^\infty(M). \tag{4.3}$$

On the other hand, the weight-one component consists of $\alpha + X \in \Omega^1(P) \oplus \mathcal{T}(P)$ such that

$$\begin{cases} (A^P + h_A)_0(\alpha + X) = L_{A^P}\alpha + [A^P, X] = 0 \\ (A^P + h_A)_1(\alpha + X) = \alpha(A^P) + \{A^P, X\} + h_A(X) = 0 \end{cases} \qquad \text{for } A \in \mathfrak{g}$$

where we have used (A.3), Lemma 2.7 and (3.11). It follows that there is a pullback square

$$\begin{tikzcd} \mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}_1 \arrow[r] \arrow[d] & \mathcal{T}(P)^G \arrow[d, "q"] \\ \Omega^1(P)^G \arrow[r, "p"] & (\mathfrak{g}^\vee \otimes C^\infty(P))^G \end{tikzcd}$$

where p, q are suitable adjoints to $(\alpha, A) \mapsto -\alpha(A^P)$ and $(X, A) \mapsto \{A^P, X\} + h_A(X)$ respectively. Observe that (i) p is surjective, as contraction with $\Theta \in (\Omega^1(P) \otimes \mathfrak{g})^G$ furnishes a right inverse, and (ii) $\ker p$ is the space of the G-invariant horizontal (i.e. basic) 1-forms on P, or in other words $\pi^*\Omega^1(M)$. This implies the following short exact sequence

$$0 \longrightarrow \pi^*\Omega^1(M) \longrightarrow \mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}_1 \longrightarrow \mathcal{T}(P)^G \longrightarrow 0 \tag{4.4}$$

In view of (4.3) and (4.4), the vertex algebroid associated to $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ (see §A.5) is of the form

$$\big(C^\infty(M), \Omega^1(M), \mathcal{T}(P)^G, \cdots\big). \tag{4.5}$$

Since $\mathcal{D}^{\mathrm{ch}}(P)$ is freely generated by its associated vertex algebroid (see §A.7), $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ must contain at least a subalgebra that is freely generated by (4.5). However, the author does not know if $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ may or may not contain more elements (see also Example 4.5).

Remarks. (i) By the above discussion, any module over $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ contains in its lowest weight a module over the Atiyah algebroid $(C^\infty(M), \mathcal{T}(P)^G)$, which can be e.g. the space of sections of a vector bundle with connection associated to $(P \to M, \Theta)$. [4] In subsequent sections we will construct and study such modules over $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$. (ii) It is noteworthy that $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$ is different from an algebra of CDOs on the base manifold M — compare (4.3)–(4.4) and (2.9) — and by comparison has more interesting modules. In §6 we will see that to recover an algebra of CDOs on M from $\mathcal{D}^{\mathrm{ch}}(P)$ requires the use of semi-infinite cohomology (as well as a suitable choice of $P \to M$ plus other geometric data).

Example 4.5. The invariant subalgebra on a Lie group. This is a continuation of Example 3.12 and notations will not be explained again. Recall the subalgebras $V_{-s-2h^\vee}(\mathfrak{g})^\ell$, $V_s(\mathfrak{g})^r \subset \mathcal{D}^{\mathrm{ch}}_s(G)$ generated respectively by $A^\ell + h^\ell_A$ and $A^r + h^r_A$ for $A \in \mathfrak{g}$. Consider the centralizer subalgebra

$$\mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell} = C\Big(\mathcal{D}^{\mathrm{ch}}_s(G), V_{-s-2h^\vee}(\mathfrak{g})^\ell\Big).$$

By the discussion in §4.4 and (3.13), $\mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell}_0 = \mathbb{C}$ and $\mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell}_1$ consists of elements of the form $A^r + \alpha$ where $A \in \mathfrak{g}$ and $\alpha \in \Omega^1(G)$ satisfies

$$\alpha(B^\ell) = -\{B^\ell, A^r\} - h^\ell_B(A^r) = \frac{s}{2}\,\lambda_0(B, \theta(A^r)) = h^r_A(B^\ell) \qquad \text{for } B \in \mathfrak{g}$$

i.e. $\alpha = h^r_A$. In other words, we have $\mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell}_i = V_s(\mathfrak{g})^r_i$ for $i = 0, 1$. Since $V_s(\mathfrak{g})^r$ is freely generated by its two lowest weights, there is an inclusion

$$V_s(\mathfrak{g})^r \subset \mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell}. \tag{4.6}$$

[4] Given a G-representation W, the space of sections of the associated vector bundle is $(C^\infty(P) \otimes W)^G$, which is naturally a module over $(C^\infty(M), \mathcal{T}(P)^G)$. In fact, the action of $\widetilde{X} \in \mathcal{T}_h(P)^G$ can be identified with the covariant derivative along the corresponding $X \in \mathcal{T}(M)$, and the action of $\mathcal{T}_v(P)^G \cong (C^\infty(P) \otimes \mathfrak{g})^G$ is induced by the representation $\mathfrak{g} \to \operatorname{End} W$.

This means the subalgebras $V_{-s-2h^\vee}(\mathfrak{g})^\ell$ and $V_s(\mathfrak{g})^r$ commute with each other.

If $s \neq -h^\vee$, then both $V_{-s-2h^\vee}(\mathfrak{g})^\ell$ and $V_s(\mathfrak{g})^r$ admit Sugawara conformal vectors [Kac98, FB04]

$$\nu^\ell \in V_{-s-2h^\vee}(\mathfrak{g})^\ell \text{ of central charge } \left(\frac{s+2h^\vee}{s+h^\vee}\right)\dim\mathfrak{g},$$
$$\nu^r \in V_s(\mathfrak{g})^r \text{ of central charge } \left(\frac{s}{s+h^\vee}\right)\dim\mathfrak{g}.$$

On the other hand, since $\mathcal{D}^{\mathrm{ch}}_s(G)$ has a conformal vector ν of central charge $2\dim\mathfrak{g}$ and the subalgebra $V_{-s-2h^\vee}(\mathfrak{g})^\ell$ is generated by elements that are primary with respect to ν, the coset construction yields a conformal vector [Kac98, FB04]

$$\nu - \nu^\ell \in \mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell} \text{ of central charge } \left(\frac{s}{s+h^\vee}\right)\dim\mathfrak{g}.$$

In fact, a computation (which we omit) shows that $\nu - \nu^\ell = \nu^r$, i.e. the Sugawara vector of $V_s(\mathfrak{g})^r$ is also conformal for $\mathcal{D}^{\mathrm{ch}}_s(G)^{\hat{\mathfrak{g}},\ell}$. This seems to suggest that the inclusion (4.6) is in fact surjective, at least for generic values of s, but the author has yet to find either a proof or a counterexample. [5]

§ 4.6. Setting refined: principal frame bundle. In addition to the data described in §4.1, suppose there is also a representation $\rho : G \to SO(\mathbb{R}^d)$ and an isomorphism $P \times_\rho \mathbb{R}^d \cong TM$. This induces a Riemannian metric on M and an orthogonal connection on TM, which (for simplicity) is assumed to be torsion-free. Let $[p, v] \in TM$ denote the coset of $(p, v) \in P \times \mathbb{R}^d$. For $i = 1, \ldots, d$, let $\mathbf{e}_i \in \mathbb{R}^d$ be the standard basis vectors and $\tau_i \in \mathcal{T}_h(P)$ the tautological vector fields defined by

$$\tau_i|_p := \text{horizontal lift of } [p, \mathbf{e}_i]$$

which constitute a framing of T_hP. For $A \in \mathfrak{g}$ and $i, j = 1, \ldots, d$, we have the Lie brackets

$$[A^P, \tau_i] = \rho(A)_{ji}\tau_j, \qquad [\tau_i, \tau_j] = -\Omega(\tau_i, \tau_j)^P \tag{4.7}$$

where ρ here denotes the induced map of Lie algebras $\mathfrak{g} \to \mathfrak{so}_d$. [6]

Suppose the conformal algebra of CDOs $\mathcal{D}^{\mathrm{ch}}(P) = \mathcal{D}^{\mathrm{ch}}_{\nabla', H'}(P)$ is defined as in §3.3 using the following data. (Notice the various change of notations here.) Let ∇' be the G-invariant *flat* connection on TP with respect to which all A^P and τ_i are parallel, and H' be a G-invariant *closed* 3-form on P; together they determine the vertex algebroid

$$\big(C^\infty(P), \Omega^1(P), \mathcal{T}(P), \bullet', \{\ \}', \{\ \}'_\Omega\big) \tag{4.8}$$

by which $\mathcal{D}^{\mathrm{ch}}(P)$ is freely generated (see Theorem 2.5a). Also, the trivial 1-form determines a conformal vector of $\mathcal{D}^{\mathrm{ch}}(P)$, namely

$$\nu' = t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i \tag{4.9}$$

where $t_1, t_2, \cdots$ are a basis of $\mathfrak{g}$; $\Theta^1, \Theta^2, \cdots$ the corresponding components of Θ, so that $\Theta = \Theta^a \otimes t_a$; $\tau_1, \cdots, \tau_d$ the vector fields defined above; and $\tau^1, \cdots, \tau^d$ the 1-forms given by $\tau^i(\tau_j) = \delta^i_j$ and $\tau^i(A^P) = 0$ (see Theorem 2.5b). In the rest of this section, we will provide a more detailed description of this conformal vertex algebra in the presence of an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action.

According to the axioms of a vertex algebroid (Definition A.3), the entire structure of (4.8) is determined by the special cases appearing in the following statement.

[5] Notice that Lemma A.12 does not apply here.

[6] $[\tau_i, \tau_j]$ has no horizontal component precisely because of our assumption that the induced connection on TM is torsion-free. Moreover, the Jacobi identity for any τ_i, τ_j, τ_k is the same as the two Bianchi identities combined.

Lemma 4.7. *For $f \in C^\infty(P)$, $A, B \in \mathfrak{g}$ and $i, j = 1, \ldots, d$, we have*

- $A^P \bullet' f = \tau_i \bullet' f = 0$
- $\{A^P, B^P\}' = -(\lambda_{\mathsf{ad}} + \lambda_\rho)(A, B), \quad \{A^P, \tau_i\}' = 0, \quad \{\tau_i, \tau_j\}' = 2\mathrm{Ric}_{ij}$
- $\{A^P, B^P\}'_\Omega = \frac{1}{2}\iota_{A^P}\iota_{B^P}H', \quad \{A^P, \tau_i\}'_\Omega = \frac{1}{2}\iota_{A^P}\iota_{\tau_i}H', \quad \{\tau_i, \tau_j\}'_\Omega = d\,\mathrm{Ric}_{ij} + \frac{1}{2}\iota_{\tau_i}\iota_{\tau_j}H'$

where $\mathrm{Ric}_{ij} = \rho(\Omega(\tau_i, \tau_k))_{jk} \in C^\infty(P)$. *Moreover, we also have* $\nu'_1 A^P = \nu'_1 \tau_i = 0$.

Proof. Recall (2.6) and (2.8). For $\bullet$ the claim is clear. Using (4.7) we compute the following operators

$$\nabla'^{\,t} A^P : \begin{cases} B^P \mapsto -[A, B]^P \\ \tau_i \ \mapsto -\rho(A)_{ji}\tau_j \end{cases}, \qquad \nabla'^{\,t}\tau_i : \begin{cases} A^P \mapsto \rho(A)_{ji}\tau_j \\ \tau_j \ \mapsto \Omega(\tau_i, \tau_j)^P \end{cases}$$

$$\Rightarrow \qquad \nabla'(\nabla'^{\,t} A^P) = 0, \qquad \nabla'(\nabla'^{\,t}\tau_i) : \begin{cases} A^P \mapsto 0 \\ \tau_j \ \mapsto (d\Omega(\tau_i, \tau_j))^P \end{cases}$$

The rest of the lemma now follows easily from these calculations and the symmetries of the Riemannian curvature tensor $\rho(\Omega)$. ☐

Theorem 3.11 specializes to our current setting as follows.

Corollary 4.8. *The G-action on P lifts to an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}(P)$ if and only if*

$$(\lambda + \lambda_{\mathsf{ad}} + \lambda_\rho)(P) = 0.$$

Moreover, this action is primary with respect to the conformal vector ν'.

Proof. By assumption, $P \to M$ is a lifting of the special orthogonal frame bundle $F_{SO}(TM) \to M$ along $\rho : G \to SO_d$. Hence we have a commutative diagram

$$\begin{tikzcd} & H^4(BSO_d) \arrow[r, "\rho^*"] \arrow[d] & H^4(BG) \arrow[d] \\ H^4(M) \arrow[r, "\cong"] & H^4_{SO_d}(F_{SO}(TM)) \arrow[r, "\cong"] & H^4_G(P) \end{tikzcd}$$

Also recall the identifications like $H^4(BG) \cong (\mathrm{Sym}^2\mathfrak{g}^\vee)^G$. Since $-8\pi^2 p_1 \in H^4(BSO_d)$ can be identified with the bilinear form $(A, B) \mapsto \mathrm{Tr}\,AB$, its image under ρ^* is λ_ρ and so by the diagram above we have $-8\pi^2 p_1(M) = \lambda_\rho(P)$. Now the first claim is just a special case of Corollary 4.3. The other claim is true because $\nu'_1 A^P = 0$ for $A \in \mathfrak{g}$ by Lemma 4.7. ☐

Remark. In the sequel we will write $\lambda^* = \lambda + \lambda_{\mathsf{ad}} + \lambda_\rho$.

§ 4.9. Inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on CDOs. Suppose $\mathcal{D}^{\mathrm{ch}}(P)$ is a principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebra, i.e. there is an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action $V_\lambda(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}(P)$, defined by some associated Cartan cochain $h \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$. Notice that the Cartan cochains in (3.2) may now be written as $\chi^{2,2} = \frac{1}{2}d_G H'$ and $\chi^{4,0} = -\lambda_{\mathsf{ad}} - \lambda_\rho$ by Lemma 4.7. Hence the condition (3.11) on h is now equivalent to the equation

$$d_G(H' - 2h) = \lambda^*. \tag{4.10}$$

This of course agrees with Corollary 4.8. The purpose of the next two lemmas is to modify the description of the vertex algebra $\mathcal{D}^{\mathrm{ch}}(P)$ in such a way that the inner $(\hat{\mathfrak{g}}_\lambda, G)$-action becomes manifest and the data (H', h) involved are replaced by a basic 3-form that trivializes $\lambda^*(\Omega \wedge \Omega)$.

Lemma 4.10. *There is a natural bijection between the following two sets of data:*
(i) $(H', h) \in \Omega^3(P)^G \oplus (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$ *such that* $d_G(H' - 2h) = \lambda^*$, *and*
(ii) $(H, \beta) \in \pi^*\Omega^3(M) \oplus \Omega^2(P)^G$ *such that* $dH = \lambda^*(\Omega \wedge \Omega)$ *and* $\beta|_{T_h P} = 0$.

Proof. Once the maps in both directions are described below, it will be clear they are inverses to each other. Let $\mathrm{CS}_{\lambda^*}(\Theta) = \lambda^*(\Theta \wedge \Omega) - \frac{1}{6}\lambda^*(\Theta \wedge [\Theta \wedge \Theta])$. Notice that as Cartan cochains $\mathrm{CS}_{\lambda^*}(\Theta) \in \Omega^3(P)^G$ and $\lambda^*(-, \Theta) \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$ have the following differentials

$$d_G \, \mathrm{CS}_{\lambda^*}(\Theta) = \lambda^*(\Omega \wedge \Omega) - \lambda^*(-, d\Theta), \qquad d_G \, \lambda^*(-, \Theta) = \lambda^*(-, d\Theta) - \lambda^*. \tag{4.11}$$

From (i) to (ii). Let (H', h) be a pair as in (i). First define a G-invariant 2-form on P by

$$\beta(A^P, B^P) = h_A(B^P) - h_B(A^P), \qquad \beta(A^P, \tau_i) = 2h_A(\tau_i), \qquad \beta(\tau_i, \tau_j) = 0 \tag{4.12}$$

for $A, B \in \mathfrak{g}$ and $i, j = 1, \dots, d$. Indeed, the G-equivariance of h together with (4.7) imply the G-invariance of β. Observe that the component of $d_G(H' - 2h) = \lambda^*$ in $(\mathrm{Sym}^2 \mathfrak{g}^\vee \otimes C^\infty(P))^G$ allows us to write

$$\iota_{A^P} \beta = 2h_A - \lambda^*(A, \Theta). \tag{4.13}$$

Then define a G-invariant 3-form on P by the first expression below

$$H = H' + \mathrm{CS}_{\lambda^*}(\Theta) - d\beta = H' + \mathrm{CS}_{\lambda^*}(\Theta) - 2h + \lambda^*(-, \Theta) - d_G\beta \tag{4.14}$$

but we also have the second expression thanks to (4.13). It follows from $d_G(H' - 2h) = \lambda^*$ and (4.11) that $d_G H = \lambda^*(\Omega \wedge \Omega)$, or equivalently, H is horizontal (hence basic) and satisfies $dH = \lambda^*(\Omega \wedge \Omega)$.

From (ii) to (i). Use (4.14) again: given (H, β) as in (ii), define the desired pair (H', h) by

$$H' - 2h = H - \mathrm{CS}_{\lambda^*}(\Theta) - \lambda^*(-, \Theta) + d_G\beta.$$

Indeed, it follows from $d_G H = \lambda^*(\Omega \wedge \Omega)$ and (4.11) that $d_G(H' - 2h) = \lambda^*$. $\square$

Preparation. This continues with the discussion in §4.9. Since the data (H', h) in the definition of $\mathcal{D}^{\mathrm{ch}}(P)$ as a principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebra satisfy (4.10), by Lemma 4.10 they give rise to a G-invariant 2-form β and a basic 3-form H satisfying $dH = \lambda^*(\Omega \wedge \Omega)$. Let $\Delta : \mathcal{T}(P) \to \Omega^1(P)$ be the $C^\infty(P)$-linear map with $\Delta(A^P) = -h_A$ for $A \in \mathfrak{g}$ and $\Delta(\tau_i) = -\frac{1}{2}\iota_{\tau_i}\beta$ for $i = 1, \dots, d$. By Lemma A.9, Δ determines an isomorphism (id, Δ) from the vertex algebroid (4.8) to a new vertex algebroid

$$\big(C^\infty(P), \Omega^1(P), \mathcal{T}(P), \bullet, \{\ \}, \{\ \}_\Omega\big). \tag{4.15}$$

This in turn induces an isomorphism from $\mathcal{D}^{\mathrm{ch}}(P)$ to the vertex algebra freely generated by (4.15), which we denote by $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$. By composition, there is an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action $V_\lambda(\mathfrak{g}) \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ whose associated Cartan cochain is trivial, i.e. it is determined by the map $\mathfrak{g} \hookrightarrow \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)_1$ sending A simply to A^P.

According to the axioms of a vertex algebroid (Definition A.3), the entire structure of (4.15) is determined by the following special cases.

Lemma 4.11. *For $f \in C^\infty(P)$, $A, B \in \mathfrak{g}$ and $i, j = 1, \dots, d$, we have*

- $A^P \bullet f = \tau_i \bullet f = 0$
- $\{A^P, B^P\} = \lambda(A, B), \quad \{A^P, \tau_i\} = 0, \quad \{\tau_i, \tau_j\} = 2\mathrm{Ric}_{ij}$
- $\{A^P, B^P\}_\Omega = \{A^P, \tau_i\}_\Omega = 0, \quad \{\tau_i, \tau_j\}_\Omega = d\,\mathrm{Ric}_{ij} + \frac{1}{2}\iota_{\tau_i}\iota_{\tau_j} H + \lambda^*(\Omega(\tau_i, \tau_j), \Theta)$

Proof. Recall Definition A.4 for the meaning of $(\mathrm{id}, \Delta) : (4.8) \to (4.15)$ as well as Lemma 4.7 for the structure maps of (4.8). Firstly, the claim for $\bullet$ follows from the $C^\infty(P)$-linearity of Δ. Consider again the map of vertex algebroid (i, h) in Proposition 3.8. Because of the composition

$$(\mathrm{id}, \Delta) \circ (i, h) = (i, 0) : (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0) \to (4.15)$$

it is clear that $\{A^P, B^P\} = \lambda(A, B)$ and $\{A^P, B^P\}_\Omega = 0$. This is equivalent to the comment above concerning the inner $(\hat{\mathfrak{g}}_\lambda, G)$-action on $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$. The other values of $\{\ \}$ are computed as follows:

$$\begin{aligned}
\{A^P, \tau_i\} &= \{A^P, \tau_i\}' + h_A(\tau_i) + \tfrac{1}{2}\beta(\tau_i, A^P) = 0 \\
\{\tau_i, \tau_j\} &= \{\tau_i, \tau_j\}' + \tfrac{1}{2}\beta(\tau_i, \tau_j) + \tfrac{1}{2}\beta(\tau_j, \tau_i) = 2\mathrm{Ric}_{ij}
\end{aligned}$$

using the definition of β in (4.12). The other values of $\{\ \}_\Omega$ are computed as follows:

$$\begin{aligned}
\{A^P, \tau_i\}_\Omega &= \{A^P, \tau_i\}'_\Omega + \tfrac{1}{2}L_{A^P}\iota_{\tau_i}\beta - L_{\tau_i}h_A + dh_A(\tau_i) - \tfrac{1}{2}\iota_{[A^P,\tau_i]}\beta \\
&= \tfrac{1}{2}\iota_{A^P}\iota_{\tau_i}H' - \iota_{\tau_i}dh_A = 0 \\
\{\tau_i, \tau_j\}_\Omega &= \{\tau_i, \tau_j\}'_\Omega + \tfrac{1}{2}L_{\tau_i}\iota_{\tau_j}\beta - \tfrac{1}{2}L_{\tau_j}\iota_{\tau_i}\beta + \tfrac{1}{2}d\beta(\tau_i, \tau_j) + h_{\Omega(\tau_i,\tau_j)} \\
&= d\,\mathrm{Ric}_{ij} + \tfrac{1}{2}\iota_{\tau_i}\iota_{\tau_j}H' - \tfrac{1}{2}\iota_{\Omega(\tau_i,\tau_j)}\beta - \tfrac{1}{2}\iota_{\tau_i}\iota_{\tau_j}d\beta + \tfrac{1}{2}\iota_{\Omega(\tau_i,\tau_j)}\beta + \tfrac{1}{2}\lambda^*(\Omega(\tau_i, \tau_j), \Theta) \\
&= d\,\mathrm{Ric}_{ij} + \tfrac{1}{2}\iota_{\tau_i}\iota_{\tau_j}H + \lambda^*(\Omega(\tau_i, \tau_j), \Theta)
\end{aligned}$$

using, in order: the G-invariance of β; the component of (4.10) in $(\mathfrak{g}^\vee \otimes \Omega^2(P))^G$; the identity

$$L_{\tau_i}\iota_{\tau_j} - L_{\tau_j}\iota_{\tau_i} + d\iota_{\tau_j}\iota_{\tau_i} = L_{\tau_i}\iota_{\tau_j} - \iota_{\tau_j}d\iota_{\tau_i} = L_{\tau_i}\iota_{\tau_j} - \iota_{\tau_j}L_{\tau_i} + \iota_{\tau_j}\iota_{\tau_i}d = \iota_{[\tau_i,\tau_j]} - \iota_{\tau_i}\iota_{\tau_j}d;$$

the Lie bracket $[\tau_i, \tau_j] = -\Omega(\tau_i, \tau_j)^P$; equation (4.13); and finally the definition of H in (4.14). □

Remark. Because of the bijection in Lemma 4.10, we know that *any* basic 3-form H that satisfies $dH = \lambda^*(\Omega \wedge \Omega)$ determines a vertex algebroid (4.15) as per Lemma 4.11, and hence equivalently a principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$.

Corollary 4.12. *In the vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ we have the following normal-ordered expansions and commutation relations* [7]

$$\begin{aligned}
&(fA^P)_n = \sum_{k\geq 0} f_{n-k}A^P_k + \sum_{k<0} A^P_k f_{n-k}, \qquad (f\tau_i)_n = \sum_{k\geq 0} f_{n-k}\tau_{i,k} + \sum_{k<0} \tau_{i,k}f_{n-k} \\
&[A^P_n, B^P_m] = [A, B]^P_{n+m} + n\lambda(A, B)\delta_{n+m,0}, \qquad [A^P_n, \tau_{i,m}] = \rho(A)_{ji}\tau_{j,n+m} \\
&[\tau_{i,n}, \tau_{j,m}] = -\Omega(\tau_i, \tau_j)^P_{n+m} + (n-m)(\mathrm{Ric}_{ij})_{n+m} + \Big(\tfrac{1}{2}\iota_{\tau_i}\iota_{\tau_j}H + \lambda^*(\Omega(\tau_i, \tau_j), \Theta)\Big)_{n+m}
\end{aligned}$$

for $f \in C^\infty(P)$, $A, B \in \mathfrak{g}$, $i, j = 1, \ldots, d$ and $n, m \in \mathbb{Z}$.

Proof. This follows immediately from Lemma 4.11 and Definition A.7. □

Lemma 4.13. *The vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ has a conformal vector*

$$\nu = t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - \frac{1}{2}\lambda^*(\Theta_{-1}\Theta)$$

of central charge $2\dim P$. Moreover, we have $\nu_1 A^P = \nu_1\tau_i = 0$ for $A \in \mathfrak{g}$ and $i = 1, \ldots, d$.

Proof. By construction the isomorphism from $\mathcal{D}^{\mathrm{ch}}(P)$ to $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ (see *Preparation* before Lemma 4.11) sends the conformal vector ν' in (4.9) to

$$\begin{aligned}
\nu &= (t^P_a - h_{t_a})_{-1}\Theta^a + (\tau_i - \tfrac{1}{2}\iota_{\tau_i}\beta)_{-1}\tau^i \\
&= t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - (h_{t_a})_{-1}\Theta^a - \tfrac{1}{2}(\iota_{\tau_i}\beta)_{-1}\tau^i \\
&= t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - \Big(h_{t_a}(t^P_b)\Theta^b + h_{t_a}(\tau_i)\tau^i\Big)_{-1}\Theta^a - \tfrac{1}{2}\Big(\beta(\tau_i, t^P_a)\Theta^a + \beta(\tau_i, \tau_j)\tau^j\Big)_{-1}\tau^i \\
&= t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - \tfrac{1}{2}\lambda^*(t_a, t_b)\Theta^b_{-1}\Theta^a - h_{t_a}(\tau_i)\tau^i_{-1}\Theta^a + h_{t_a}(\tau_i)\tau^i_{-1}\Theta^a + 0 \\
&= t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - \tfrac{1}{2}\lambda^*(\Theta_{-1}\Theta)
\end{aligned}$$

[7] The author found it interesting (and comforting) to verify directly all Jacobi identities in sight.

where we have used, in order: the component of (4.10) in $(\mathrm{Sym}^2\mathfrak{g}^\vee \otimes C^\infty(P))^G$; the symmetry $\alpha_{-1}\alpha' = \alpha'_{-1}\alpha$ for 1-forms α, α'; and the definition of β in (4.12). The said isomorphism also sends $\nu'_1(A^P + h_A)$ to $\nu_1 A^P$ and $\nu'_1(\tau_i + \iota_{\tau_i}\beta)$ to $\nu_1\tau_i$. This proves our claims in view of Lemma 4.7 and (2.8). $\square$

The following summarizes all our discussions since §4.6.

Theorem 4.14. *Suppose we have the following data: a principal G-bundle $\pi : P \to M$, a representation $\rho : G \to SO(\mathbb{R}^d)$ and an isomorphism $P \times_\rho \mathbb{R}^d \cong TM$; a connection Θ on π that induces the Levi-Civita connection on TM; an invariant symmetric bilinear form λ on $\mathfrak{g}$; and a basic 3-form H on P satisfying $dH = \lambda^*(\Omega \wedge \Omega)$, where Ω is the curvature of Θ and $\lambda^* = \lambda + \lambda_{\mathsf{ad}} + \lambda_\rho$.*

Given these data, there is an associated vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ with generating fields

$$\mathsf{Y}(f,x) \text{ for } f \in C^\infty(P), \qquad \mathsf{Y}(A^P,x) \text{ for } A \in \mathfrak{g}, \qquad \mathsf{Y}(\tau_i,x) \text{ for } i = 1,\ldots,d$$

whose weights are 0, 1 and 1 respectively. (For the notations see §4.1 and §4.6.) Notice that the vacuum is $1 \in C^\infty(P)$. The various OPEs with $\mathsf{Y}(f,x)$ have the following leading terms

$$\begin{aligned}
\mathsf{Y}(g,x')\mathsf{Y}(f,x) &= \mathsf{Y}(fg,x) + \mathsf{Y}(fdg,x)(x'-x) + O((x'-x)^2), \quad g \in C^\infty(P)\\
\mathsf{Y}(A^P,x')\mathsf{Y}(f,x) &= \frac{\mathsf{Y}(A^P f,x)}{x'-x} + \mathsf{Y}(fA^P,x) + O(x'-x)\\
\mathsf{Y}(\tau_i,x')\mathsf{Y}(f,x) &= \frac{\mathsf{Y}(\tau_i f,x)}{x'-x} + \mathsf{Y}(f\tau_i,x) + O(x'-x)
\end{aligned}$$

where in fact all the second terms are meant to be taken (plus linearity) as the definition of fields associated to arbitrary 1-forms and vector fields on P. The OPEs between the other generating fields have the following singular parts

$$\mathsf{Y}(A^P,x')\mathsf{Y}(B^P,x) \sim \frac{\lambda(A,B)}{(x'-x)^2} + \frac{\mathsf{Y}([A,B]^P,x)}{x'-x}, \qquad \mathsf{Y}(A^P,x')\mathsf{Y}(\tau_i,x) \sim \frac{\rho(A)_{ji}\mathsf{Y}(\tau_j,x)}{x'-x}$$
$$\mathsf{Y}(\tau_i,x')\mathsf{Y}(\tau_j,x) \sim \frac{2\mathsf{Y}\big(\mathrm{Ric}_{ij}, \frac{x+x'}{2}\big)}{(x'-x)^2} - \frac{\mathsf{Y}\big(\Omega(\tau_i,\tau_j)^P,x\big)}{x'-x} + \frac{\frac{1}{2}\mathsf{Y}\big(\iota_{\tau_i}\iota_{\tau_j}H,x\big) + \mathsf{Y}\big(\lambda^*\big(\Omega(\tau_i,\tau_j),\Theta\big),x\big)}{x'-x}$$

where $\mathrm{Ric}_{ij} = \rho(\Omega(\tau_i,\tau_k))_{jk}$. Clearly the fields $\mathsf{Y}(A^P,x)$ for $A \in \mathfrak{g}$ represent an inner $(\hat{\mathfrak{g}}_\lambda, G)$-action (see Definition 3.4). Moreover, there is a Virasoro field of central charge $2\dim P$ given by

$$:\mathsf{Y}(t_a^P,x)\mathsf{Y}(\Theta^a,x): + :\mathsf{Y}(\tau_i,x)\mathsf{Y}(\tau^i,x): - \frac{1}{2}\lambda^*\big(\mathsf{Y}(\Theta,x),\mathsf{Y}(\Theta,x)\big)$$

with respect to which all generating fields are primary. (For the notations see §4.6.) $\square$

Remark. For convenience (and lack of imagination), we will refer to $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ as a **principal frame $(\hat{\mathfrak{g}}_\lambda, G)$-algebra** or, if we do not wish to specify λ, a **principal frame $(\hat{\mathfrak{g}}, G)$-algebra**. Even though this vertex algebra arises here as a particular algebra of CDOs, it should really be regarded as a more fundamental object for both conceptual and aesthetic reasons. In §6 we will see that any algebra of CDOs can be recovered in a natural way from a principal frame $(\hat{\mathfrak{g}}_\lambda, G)$-algebra with $\lambda = -\lambda_{\mathsf{ad}}$; and this is only a special case of a more general construction associated to principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebras with different λ (to be introduced in §5). Also noteworthy is that, compared to an arbitrary algebra of CDOs, a principal frame $(\hat{\mathfrak{g}}, G)$-algebra requires a smaller set of generating fields and possesses arguably more appealing OPEs and Virasoro field. [8] For these reasons, it seems desirable to find a more direct construction of principal frame $(\hat{\mathfrak{g}}, G)$-algebras, perhaps in a future paper.

[8] For generating fields, $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ requires only a finite number of canonical vector fields (that constitute a framing of TP), as opposed to all vector fields. For OPEs, compare the vertex algebroids in Lemma 4.11 and Theorem 2.5a. For Virasoro fields, compare the conformal vectors in Lemma 4.13 and Theorem 2.5b.

§ 4.15. The invariant subalgebra. This discussion is a more detailed version of §4.4 specifically for a principal frame $(\hat{\mathfrak{g}}_\lambda, G)$-algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$. Consider the centralizer subalgebra

$$\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\hat{\mathfrak{g}}} := C\big(\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P), V_\lambda(\mathfrak{g})\big).$$

The weight-zero component is again $C^\infty(P)^G$. Let us describe the weight-one component below.

Consider an arbitrary element of $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\hat{\mathfrak{g}}}_1$: some $\alpha + \mathscr{X} + \mathscr{Y} \in \Omega^1(P) \oplus \mathcal{T}_h(P) \oplus \mathcal{T}_v(P)$ that is annihilated by A^P_0 and A^P_1 for $A \in \mathfrak{g}$. Suppose $t_1, t_2, \cdots$ are a basis of $\mathfrak{g}$ and $t^1, t^2, \cdots$ the dual basis of $\mathfrak{g}^\vee$. Let us write $\mathscr{X} = \mathscr{X}^i \tau_i$ and $\mathscr{Y} = \mathscr{Y}^a t^P_a$. By Lemma 4.11 and Definition A.3, both $\{A^P, \mathscr{X}^i\tau_i\}_\Omega$ and $\{A^P, \mathscr{Y}^a t^P_a\}_\Omega$ vanish; then by (A.3) the first condition on $\alpha + \mathscr{X} + \mathscr{Y}$ reads

$$A^P_0(\alpha + \mathscr{X} + \mathscr{Y}) = L_{A^P}\alpha + [A^P, \mathscr{X}] + [A^P, \mathscr{Y}] = 0 \qquad \text{for } A \in \mathfrak{g}.$$

Hence α, $\mathscr{X}$, $\mathscr{Y}$ are all G-invariant. Notice that the G-invariance of $\mathscr{Y}$ means $A^P\mathscr{Y}^a = -t^a([A, t_b])\mathscr{Y}^b$. By Lemma 4.11 and Definition A.3 again, we find that $\{A^P, \mathscr{X}^i\tau_i\} = \rho(A)_{ji}\tau_j\mathscr{X}^i$ and

$$\begin{aligned}
\{A^P, \mathscr{Y}^a t^P_a\} &= \lambda(A, t_a)\mathscr{Y}^a + [A, t_a]^P\mathscr{Y}^a \\
&= \lambda(A, t_a)\mathscr{Y}^a - t^a([[A, t_a], t_b])\mathscr{Y}^b \\
&= (\lambda + \lambda_{\mathsf{ad}})(A, \mathscr{Y});
\end{aligned}$$

then by (A.3) the second condition on $\alpha + \mathscr{X} + \mathscr{Y}$ reads

$$A^P_1(\alpha + \mathscr{X} + \mathscr{Y}) = \alpha(A^P) + \rho(A)_{ij}\tau_i\mathscr{X}^j + (\lambda + \lambda_{\mathsf{ad}})(A, \mathscr{Y}) = 0 \qquad \text{for } A \in \mathfrak{g}.$$

It follows from the two conditions that the 1-form $\alpha + (\tau_i\mathscr{X}^j)\rho(\Theta)_{ij} + (\lambda + \lambda_{\mathsf{ad}})(\Theta, \mathscr{Y})$ is G-invariant and horizontal (i.e. basic). Therefore $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\hat{\mathfrak{g}}}_1$ is the direct sum of the following subspaces:

$$\pi^*\Omega^1(M), \quad \big\{\mathscr{X} - (\tau_i\mathscr{X}^j)\rho(\Theta)_{ij} : \mathscr{X} \in \mathcal{T}_h(P)^G\big\}, \quad \big\{\mathscr{Y} - (\lambda + \lambda_{\mathsf{ad}})(\Theta, \mathscr{Y}) : \mathscr{Y} \in \mathcal{T}_v(P)^G\big\}. \tag{4.16}$$

This provides in our current setting an explicit description of the short exact sequence (4.4) together with a splitting.

§5. Associated Modules and Algebras over CDOs

Consider an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}(P)$ on the total space of a principal bundle $P \to M$, equipped with a projective "formal loop group action". This section introduces a construction of modules over the invariant subalgebra of $\mathcal{D}^{\mathrm{ch}}(P)$ using semi-infinite cohomology; it is very much analogous to the construction of associated vector bundles of $P \to M$. In fact, the said modules should have an interpretation as the spaces of sections of certain vector bundles over the hypothetical "formal loop space" of M. Examples will be studied in §6 and §7.

The semi-infinite cohomology of a centrally extended loop algebra can be defined using the Feigin complex, which is a vertex algebraic analogue of the Chevalley-Eilenberg complex, but with some important differences. For more about semi-infinite cohomology, see e.g. [Fei84, Vor93, BD04].

§ 5.1. The Chevalley-Eilenberg complex. Given a Lie algebra $\mathfrak{g}$, consider the supermanifold $\Pi\mathfrak{g}$. Let $t_1, t_2, \ldots$ be a basis of $\mathfrak{g}$; $t^1, t^2, \ldots$ the dual basis of $\mathfrak{g}^\vee$; and $(\phi^1, \phi^2, \ldots)$ the corresponding coordinates of $\Pi\mathfrak{g}$. By definition $\mathcal{O}(\Pi\mathfrak{g}) = \wedge^*\mathfrak{g}^\vee$ and $\mathcal{T}(\Pi\mathfrak{g})$ consists of the derivations on $\wedge^*\mathfrak{g}^\vee$. Consider

$$J,\, q \in \mathcal{T}(\Pi\mathfrak{g}), \qquad \theta \in \mathcal{O}(\Pi\mathfrak{g}) \otimes \mathfrak{g}$$

corresponding respectively to the exterior degree on $\wedge^*\mathfrak{g}^\vee$, the Chevalley-Eilenberg differential on $\wedge^*\mathfrak{g}^\vee$, and the Maurer-Cartan form on $\mathfrak{g}$; they can be written in coordinates as follows

$$J = \phi^a \frac{\partial}{\partial \phi^a}\,, \qquad q = -\frac{1}{2}\, t^a([t_b, t_c])\phi^b\phi^c \frac{\partial}{\partial \phi^a}\,, \qquad \theta = \phi^a \otimes t_a. \tag{5.1}$$

Notice that $J = J \otimes 1$, $q = q \otimes 1$ and θ satisfy

$$[J, q] = q, \qquad J\theta = \theta, \qquad [q, q] = 0, \qquad q\theta + \frac{1}{2}\,[\theta, \theta] = 0. \tag{5.2}$$

Let W be a $\mathfrak{g}$-module. If J, q, θ are regarded as operators on $\mathcal{O}(\Pi\mathfrak{g}) \otimes W$ and $Q = q + \theta$, then by (5.2) they satisfy $[J, Q] = Q$ and $[Q, Q] = 2Q^2 = 0$. The Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in W can be written as

$$\Big(\mathcal{O}(\Pi\mathfrak{g}) \otimes W,\, J,\, Q\Big)$$

where J is the grading operator and Q is the differential.

§ 5.2. The Feigin complex. Consider the algebra of CDOs $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})$, which is a fermionic version of §2.1; for details see [Che11]. Given an invariant symmetric bilinear form λ on $\mathfrak{g}$, let $\hat{\mathfrak{g}}_\lambda$ be the centrally extended loop algebra described in §3.1 and $V_\lambda(\mathfrak{g})$ the vertex algebra in Example A.10. Now regard

$$J, q, \theta \text{ as elements of } \mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes V_\lambda(\mathfrak{g}) \text{ of weight } 1.$$

The following computations in $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes V_\lambda(\mathfrak{g})$ make use of (A.3), (5.1) and the super version of (2.4):

$$\begin{aligned}
&J_0 q = [J, q] + \{J, q\}_\Omega = q + 0 = q \\
&J_0 \theta = (J_0 \otimes 1)(\phi^a \otimes t_a) = J\phi^a \otimes t_a = \phi^a \otimes t_a = \theta \\
&q_0 q = [q, q] + \{q, q\}_\Omega = 0 - t^a([t_b, t_c])\, t^b([t_a, t_d])\, \phi^d d\phi^c = -\lambda_{\mathsf{ad}}(t_c, t_d)\, \phi^d d\phi^c \\
&\left.\begin{aligned} q_0\theta &= (q_0 \otimes 1)(\phi^a \otimes t_a) \\ \theta_0 q &= (\phi^a_1 \otimes t_{a,-1})(q \otimes 1) \end{aligned}\right\} = q\phi^a \otimes t_a = -\frac{1}{2}\phi^b\phi^c \otimes [t_b, t_c] \\
&\theta_0\theta = (\phi^a_0 \otimes t_{a,0} + \phi^a_{-1} \otimes t_{a,1})(\phi^b \otimes t_b) = \phi^a\phi^b \otimes [t_a, t_b] - \lambda(t_a, t_b)\, \phi^b d\phi^a
\end{aligned}$$

where λ_{ad} denotes the Killing form. Hence J and $Q = q + \theta$ satisfy

$$\begin{aligned}
&J_0 Q = Q, && Q_0 Q = -(\lambda_{\mathsf{ad}} + \lambda)(t_a, t_b)\, \phi^a d\phi^b \\
\Rightarrow\quad &[J_0, Q_0] = Q_0, && [Q_0, Q_0] = (\lambda_{\mathsf{ad}} + \lambda)(t_a, t_b) \cdot \textstyle\sum_{n\in\mathbb{Z}} n\phi^a_{-n}\phi^b_n
\end{aligned}$$

where we have used (A.3) and (A.4). In particular, $Q_0^2 = 0$ if and only if $\lambda = -\lambda_{\mathsf{ad}}$.

Let W be a $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-module. Notice that $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W$ is a module over $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes V_{-\lambda_{\mathsf{ad}}}(\mathfrak{g})$. The Feigin complex of $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$ with coefficients in W is

$$\Big(\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W,\, J_0,\, Q_0\Big)$$

where J_0 is the grading operator and Q_0 is the differential.

§ 5.3. Semi-infinite cohomology. For any $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-module W as above, let

$$H^{\frac{\infty}{2}+*}(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, W) := H^*\Big(\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W,\, Q_0\Big). \tag{5.3}$$

Notice that if W is a vertex algebra such that the $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-action is inner, i.e. induced by a map of vertex algebras $V_{-\lambda_{\mathsf{ad}}}(\mathfrak{g}) \to W$, then (5.3) has the structure of a $\mathbb{Z}$-graded vertex algebra. (In this case, J and Q will also denote their own images in $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W$.)

Remarks. (i) Unlike Lie algebra cohomology, the grading on semi-infinite cohomology is neither bounded above nor below. More precisely, the restriction of J_0 to $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})_k \otimes W$ takes values between $-k$ and $\dim\mathfrak{g} + k$. (ii) This is only a special case of semi-infinite cohomology. For a modern exposition of the notion in the more general setting of Tate Lie algebras, see [BD04].

Lemma 5.4. *Let W be a $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-module.*

(a) The $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-invariant operators on W induce grading-preserving operators on (5.3). Moreover, if W is a vertex algebra such that its $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-action is inner, then there is a map of vertex algebras from the centralizer subalgebra $W^{\hat{\mathfrak{g}}} = C\big(W, V_{-\lambda_{\mathsf{ad}}}(\mathfrak{g})\big)$ [Kac98, FB04] to the zeroth gradation of (5.3).

(b) If there is a Virasoro action on W of central charge c such that the $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-action is primary, then it induces a grading-preserving Virasoro action on (5.3) of central charge $c - 2\dim\mathfrak{g}$. Moreover, if W is in fact a conformal vertex algebra such that its $\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}$-action is inner and primary, then (5.3) is also a conformal vertex algebra with a conformal vector in the zeroth gradation.

Proof. (a) For the first statement, suppose $F \in \operatorname{End} W$ satisfies $[A_n, F] = 0$ for $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Since on $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W$ we have

$$[Q_0, 1 \otimes F] = [q_0 \otimes 1 + \phi^a_{-n} \otimes t_{a,n},\, 1 \otimes F] = \phi^a_{-n} \otimes [t_{a,n}, F] = 0$$

the operator $1 \otimes F$ is well-defined on (5.3). Clearly it preserves the grading. For the second statement, let $u \in W^{\hat{\mathfrak{g}}}$, i.e. $u \in W$ with $A_n u = 0$ for $A \in \mathfrak{g}$ and $n \geq 0$. Since in $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W$ we have

$$Q_0(\mathbf{1} \otimes u) = (q_0 \otimes 1 + \phi^a_{-n} \otimes t_{a,n})(\mathbf{1} \otimes u) = \sum_{n \geq 0} \phi^a_{-n}\mathbf{1} \otimes t_{a,n} u = 0$$

the element $\mathbf{1} \otimes u$ represents a class $[\mathbf{1} \otimes u]$ in (5.3). Clearly $u \mapsto [\mathbf{1} \otimes u]$ is a map of vertex algebras and the image is contained in the zeroth gradation.

(b) The graded vertex algebra $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})$ has a conformal vector $\nu^{\Pi\mathfrak{g}}$ of central charge $-2\dim\mathfrak{g}$ (see §6.1); denote its Virasoro operators by $L_n^{\Pi\mathfrak{g}}$, $n \in \mathbb{Z}$. Since q and ϕ^a are primary, we have

$$[L_n^{\Pi\mathfrak{g}}, q_0] = 0, \qquad [L_n^{\Pi\mathfrak{g}}, \phi^a_m] = -(n+m)\phi^a_{n+m}, \qquad n, m \in \mathbb{Z}.$$

Since $\nu^{\Pi\mathfrak{g}}$ belongs to the zeroth gradation, all $L_n^{\Pi\mathfrak{g}}$ preserve the grading.

For the first statement, suppose $L_n^W \in \operatorname{End} W$ for $n \in \mathbb{Z}$ define a Virasoro action of central charge c and satisfy $[L_n^W, A_m] = -mA_{n+m}$ for $A \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$. Then $L_n^{\Pi\mathfrak{g}} \otimes 1 + 1 \otimes L_n^W$ for $n \in \mathbb{Z}$ define a grading-preserving Virasoro action on $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W$ of central charge $c - 2\dim\mathfrak{g}$. Since

$$\begin{aligned}
[Q_0,\, L_n^{\Pi\mathfrak{g}} \otimes 1 + 1 \otimes L_n^W] &= [q_0, L_n^{\Pi\mathfrak{g}}] \otimes 1 + [\phi^a_{-m}, L_n^{\Pi\mathfrak{g}}] \otimes t_{a,m} + \phi^a_{-m} \otimes [t_{a,m}, L_n^W] \\
&= 0 + (n-m)\phi^a_{n-m} \otimes t_{a,m} + m\phi^a_{-m} \otimes t_{a,n+m} \\
&= 0
\end{aligned}$$

the said Virasoro action is well-defined on (5.3) as well.

For the second statement, suppose $\nu^W \in W$ is a conformal vector whose Virasoro operators L^W_n, $n \in \mathbb{Z}$, satisfy $[L^W_n, A_m] = -mA_{n+m}$ for $A \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$. Then $\nu^{\Pi\mathfrak{g}} \otimes \mathbf{1} + \mathbf{1} \otimes \nu^W$ is a conformal vector of $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes W$ belonging to the zeroth gradation. Since we have

$$\begin{aligned} Q_0(\nu^{\Pi\mathfrak{g}} \otimes \mathbf{1} + \mathbf{1} \otimes \nu^W) &= Q_0(L^{\Pi\mathfrak{g}}_{-2} \otimes 1 + 1 \otimes L^W_{-2})(\mathbf{1} \otimes \mathbf{1}) \\ &= \big[Q_0,\, L^{\Pi\mathfrak{g}}_{-2} \otimes 1 + 1 \otimes L^W_{-2}\big](\mathbf{1} \otimes \mathbf{1}) \\ &= 0 \end{aligned}$$

by the computation above, $\nu^{\Pi\mathfrak{g}} \otimes \mathbf{1} + \mathbf{1} \otimes \nu^W$ also represents a conformal vector of (5.3). □

The main purpose of this section is to introduce the following construction. As before, G is a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Recall Definition 3.4 and Corollary 4.3.

Definition 5.5. Let $\pi : P \to M$ be a smooth principal G-bundle and $\lambda, \lambda' \in (\mathrm{Sym}^2\mathfrak{g}^\vee)^G$. Assume that $\lambda + \lambda' = -\lambda_{\mathsf{ad}}$ and $(\lambda + \lambda_{\mathsf{ad}})(P) = 8\pi^2 p_1(M)$. For any principal $(\hat{\mathfrak{g}}_\lambda, G)$-algebra $\mathcal{D}^{\mathrm{ch}}(P)$ and positive-energy $(\hat{\mathfrak{g}}_{\lambda'}, G)$-module W, we define

$$\Gamma^{\mathrm{ch}}(\pi, W) := H^{\frac{\infty}{2}+0}\big(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, \mathcal{D}^{\mathrm{ch}}(P) \otimes W\big).$$

Notice that if W is a vertex algebra such that the $(\hat{\mathfrak{g}}_{\lambda'}, G)$-action is inner, then $\Gamma^{\mathrm{ch}}(\pi, W)$ also has the structure of a vertex algebra (see §5.3).

Remarks. (i) The positive-energy condition means that there is a diagonalizable operator L^W_0 on W such that $[L^W_0, A_n] = -nA_n$ for $A \in \mathfrak{g}$, $n \in \mathbb{Z}$, and the eigenvalues of L^W_0 are bounded below. Whenever W admits a Virasoro action, it will be understood that L^W_0 coincides with the Virasoro operator that is usually given the same notation. For consistency, the eigenvalues of L^W_0 will also be called weights. Let $L^{\Pi\mathfrak{g}}_0$ (resp. L^P_0) be the weight operator on $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})$ (resp. $\mathcal{D}^{\mathrm{ch}}(P)$). Notice that $L^{\Pi\mathfrak{g}}_0 + L^P_0 + L^W_0$ commutes with Q_0 (see the proof of Lemma 5.4b), so that it induces a weight decomposition of $\Gamma^{\mathrm{ch}}(\pi, W)$.

(ii) The above definition still makes sense without the integrability and/or the positive-energy conditions. On the other hand, under these two conditions, we can say that the lowest-weight component W_0 of W is a G-representation and then the lowest-weight component of $\Gamma^{\mathrm{ch}}(\pi, W)$ is $(C^\infty(P) \otimes W_0)^G$, i.e. the space of sections of the associated vector bundle $P \times_G W_0 \to M$.

Lemma 5.6. *Consider again the data in Definition 5.5.*

(a) For any W as described, $\Gamma^{\mathrm{ch}}(\pi, W)$ is a module over $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}}$. Moreover, if W is a vertex algebra such that its $(\hat{\mathfrak{g}}_{\lambda'}, G)$-action is inner, then there is a map of vertex algebras $\mathcal{D}^{\mathrm{ch}}(P)^{\hat{\mathfrak{g}}} \to \Gamma^{\mathrm{ch}}(\pi, W)$.

(b) If there is a Virasoro action on W of central charge c such that the $(\hat{\mathfrak{g}}_{\lambda'}, G)$-action is primary, then it induces a Virasoro action on $\Gamma^{\mathrm{ch}}(\pi, W)$ of central charge $c + 2\dim M$. Moreover, if W is in fact a conformal vertex algebra such that its $(\hat{\mathfrak{g}}_{\lambda'}, G)$-action is inner and primary, then $\Gamma^{\mathrm{ch}}(\pi, W)$ is also a conformal vertex algebra.

Proof. Recall §4.4. This follows from Lemma 5.4, together with the fact that $\mathcal{D}^{\mathrm{ch}}(P)$ is a conformal vertex algebra with central charge $2\dim P$ and a primary inner $(\hat{\mathfrak{g}}_\lambda, G)$-action (see Theorem 3.11). □

§6. Example: Recovering Algebras of CDOs

Suppose $P \to M$ is a principal frame bundle and $\mathcal{D}^{\mathrm{ch}}(P)$ is an algebra of CDOs on P with a suitable "formal loop group action". In this section, we analyze the zeroth semi-infinite cohomology of $\mathcal{D}^{\mathrm{ch}}(P)$ and in the end identify it as an algebra of CDOs on M. This provides a description of an arbitrary algebra of CDOs that is conceptually more satisfying than the original one given in §2.

§ 6.1. Goal: a new description of algebras of CDOs. Consider the special case of Definition 5.5 associated to a principal frame $(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, G)$-algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ and the trivial $(\hat{\mathfrak{g}}_0, G)$-module $\mathbb{C}$:

$$\Gamma^{\mathrm{ch}}(\pi, \mathbb{C}) = H^{\frac{\infty}{2}+0}\big(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\big) = H^0\Big(\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g}) \otimes \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P), Q_0\Big). \tag{6.1}$$

By Lemma 5.6, $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ is a conformal vertex algebra of central charge $2\dim M$. For convenience, let us elaborate on all the data involved and recall some relevant notations.

- Let $(t_1, t_2, \cdots)$ be a basis of $\mathfrak{g}$, $(t^1, t^2, \cdots)$ the dual basis of $\mathfrak{g}^\vee$, $(\phi^1, \phi^2, \cdots)$ the corresponding coordinates of the supermanifold $\Pi\mathfrak{g}$, and $(\partial_1, \partial_2, \cdots)$ their coordinate vector fields.
- For the detailed definition of the vertex superalgebra $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})$, see [Kac98] or [Che11]. Let us mention that, as a fermionic analogue of §2.1, it is generated by such elements as $\phi^1, \phi^2, \cdots$ and $\partial_1, \partial_2, \cdots$, and has a conformal vector
$$\nu^{\Pi\mathfrak{g}} = -\partial_{a,-1} d\phi^a \tag{6.2}$$
of central charge $-2\dim\mathfrak{g}$.
- For the detailed definition of the vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$, see Theorem 4.14 with $\lambda = -\lambda_{\mathsf{ad}}$ in mind. Let us mention that it is defined using the following data: a smooth principal G-bundle $\pi : P \to M$ together with a representation $\rho : G \to SO(\mathbb{R}^d)$ and an isomorphism $P \times_\rho \mathbb{R}^d \cong TM$; a connection $\Theta = \Theta^a \otimes t_a$ on π that induces the Levi-Civita connection on TM; and a basic 3-form H on P that satisfies $dH = \lambda_\rho(\Omega \wedge \Omega)$, where $\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$. Also, it has a conformal vector
$$\nu^P = t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - \frac{1}{2}\lambda_\rho(\Theta_{-1}\Theta)$$
of central charge $2\dim P$. (As before, ρ also denotes the induced map of Lie algebras $\mathfrak{g} \to \mathfrak{so}_d$ and λ_ρ the invariant symmetric bilinear form on $\mathfrak{g}$ given by $\lambda_\rho(A, B) = \operatorname{Tr}\rho(A)\rho(B)$. For the meaning of the other notations, see §4.6.)
- For details of the Feigin complex appearing in (6.1), see §5.2. Let us mention that the grading operator J_0 is determined by $J_0(\phi^a \otimes u) = \phi^a \otimes u$ and $J_0(\partial_a \otimes u) = -\partial_a \otimes u$ for any u; and the differential $Q_0 = (q + \theta)_0$ is induced by the elements
$$q = q \otimes \mathbf{1} = -\frac{1}{2} t^a([t_b, t_c])\phi^b\phi^c\partial_a \otimes \mathbf{1}, \qquad \theta = \phi^a \otimes t^P_a. \tag{6.3}$$

  The conformal vector $\nu^{\Pi\mathfrak{g}} \otimes \mathbf{1} + \mathbf{1} \otimes \nu^P$ belongs to the zeroth gradation and is Q_0-closed.

By the assumption on π, M is an oriented Riemannian manifold. Let ∇ be the Levi-Civita connection and R the Riemannian curvature of M. The assumption on H can then be written as $dH = \operatorname{Tr}(R \wedge R)$. By Theorem 2.5, the data (∇, H) determine an algebra of CDOs $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$ with a conformal vector $\nu = \nu^0$ of central charge $2\dim M$. The goal of this section is to prove that

$$\Gamma^{\mathrm{ch}}(\pi, \mathbb{C}) = H^{\frac{\infty}{2}+0}\big(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\big) \cong \mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$$

as conformal vertex algebras. In fact, we will analyze $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ without using any prior knowledge of $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$, and effectively rediscover the latter in the end.

Throughout this section we identify $\Omega^*(M)$ with the basic subspace of $\Omega^*(P)$.

§ 6.2. The component of weight zero. Consider the Feigin complex that appears in (6.1):

$$\Big(\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})\otimes\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P),\,Q_0\Big).\tag{6.4}$$

Since its weight-zero component is simply the Chevalley-Eilenberg complex (see §5.1)

$$\Big(\mathcal{O}(\Pi\mathfrak{g})\otimes C^\infty(P),\,Q_0\Big)$$

the weight-zero component of (6.1) is

$$\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})_0=H^0\big(\mathfrak{g},C^\infty(P)\big)=C^\infty(P)^G=C^\infty(M).\tag{6.5}$$

Understanding the rest of $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ requires more work, mostly because in higher weights the Feigin complex (6.4) contains elements in both positive and negative gradations.

Lemma 6.3. *Let $\gamma=\partial_a\otimes\Theta^a\in\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})_1\otimes\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)_1$. Then we have*

$$Q_0\gamma=\nu^{\Pi\mathfrak{g}}\otimes\mathbf{1}+\mathbf{1}\otimes t^P_{a,-1}\Theta^a,\qquad\gamma_0\gamma=0$$

and therefore the anticommutators

$$[Q_0,\gamma_0]=L_0^{\Pi\mathfrak{g}}\otimes 1+1\otimes(t^P_{a,-1}\Theta^a)_0,\qquad[\gamma_0,\gamma_0]=2\gamma_0^2=0.$$

Proof. Recall the element $Q=q+\theta$ from (6.3). Let us first write

$$Q_0\gamma=\big(q_0\otimes 1+\phi^a_{-1}\otimes t^P_{a,1}+\phi^a_0\otimes t^P_{a,0}+\phi^a_1\otimes t^P_{a,-1}\big)(\partial_b\otimes\Theta^b).$$

Then each of these terms is computed as follows:

$$\begin{aligned}
&q_0\partial_b\otimes\Theta^b=\big([q,\partial_b]+\{q,\partial_b\}_\Omega\big)\otimes\Theta^b=-t^c([t_b,t_d])\phi^d\partial_c\otimes\Theta^b\\
&\phi^a_{-1}\partial_b\otimes t^P_{a,1}\Theta^b=-\partial_{b,-1}d\phi^a\otimes\Theta^b(t^P_a)=-\partial_{a,-1}d\phi^a\otimes\mathbf{1}\\
&\phi^a_0\partial_b\otimes t^P_{a,0}\Theta^b=-\partial_{b,-1}\phi^a\otimes L_{t^P_a}\Theta^b=-\phi^a\partial_b\otimes t^b([t_a,t_c])\Theta^c\\
&\phi^a_1\partial_b\otimes t^P_{a,-1}\Theta^b=\mathbf{1}\otimes t^P_{a,-1}\Theta^a
\end{aligned}$$

where we have used (A.3), the super version of (2.2)–(2.4) and the G-invariance of $\Theta=\Theta^a\otimes t_a$. In view of (6.2), this proves the claimed expression for $Q_0\gamma$. On the other hand, we have

$$\gamma_0\gamma=(\partial_{a,1}\otimes\Theta^a_{-1})(\partial_b\otimes\Theta^b)=\partial_{a,1}\partial_b\otimes\Theta^a_{-1}\Theta^b=0$$

thanks to the super version of (2.2). □

Lemma 6.4. *The weight-one component of the Feigin complex (6.4) is quasi-isomorphic to the following (Chevalley-Eilenberg) subcomplex*

$$\Big(\mathcal{O}(\Pi\mathfrak{g})\otimes\mathcal{D}^{\mathrm{ch}}_h(P)_1,\,Q_0\Big)$$

where $\mathcal{D}^{\mathrm{ch}}_h(P)_1\subset\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)_1$ is the direct sum of the subspaces

$$\Omega^1_h(P)\quad\textit{and}\quad\Big\{\mathscr{X}-(\tau_i\mathscr{X}^j)\rho(\Theta)_{ij}:\mathscr{X}=\mathscr{X}^i\tau_i\in\mathcal{T}_h(P)\Big\}.$$

Proof. First we compute the operator $(t^P_{a,-1}\Theta^a)_0$ on $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)_k$, $k=0,1$. Notice that (A.3)–(A.4) will be used repeatedly. For $f\in C^\infty(P)$ and $\alpha\in\Omega^1(P)$, we have

$$
\begin{aligned}
(t^P_{a,-1}\Theta^a)_0 f &= \Theta^a_0\, t^P_{a,0} f = 0\\
(t^P_{a,-1}\Theta^a)_0 \alpha &= \big(\Theta^a_0\, t^P_{a,0} + \Theta^a_{-1} t^P_{a,1}\big)\alpha = 0 + \alpha(t^P_a)\Theta^a = \alpha_v
\end{aligned}
$$

where α_v means the vertical part of α. For $\mathscr{X} = \mathscr{X}^i\tau_i \in \mathcal{T}_h(P)$, we have

$$
\begin{aligned}
(t^P_{a,-1}\Theta^a)_0 \mathscr{X} &= \big(t^P_{a,-1}\Theta^a_1 + \Theta^a_0\, t^P_{a,0} + \Theta^a_{-1} t^P_{a,1}\big)\mathscr{X}\\
&= 0 - \iota_{[t^P_a,\mathscr{X}]} d\Theta^a + \rho(t_a)_{ij}(\tau_i\mathscr{X}^j)\Theta^a\\
&= -\iota_{[t^P_a,\mathscr{X}]}\Omega^a + (\tau_i\mathscr{X}^j)\rho(\Theta)_{ij}
\end{aligned}
$$

using Lemma 2.7 and the computation of $\{A^P,\mathscr{X}\}$ in §4.15. For $\mathscr{Y} = \mathscr{Y}^a t^P_a \in \mathcal{T}_v(P)$, we have

$$
\begin{aligned}
(t^P_{a,-1}\Theta^a)_0 \mathscr{Y} &= \big(t^P_{a,-1}\Theta^a_1 + \Theta^a_0\, t^P_{a,0} + \Theta^a_{-1} t^P_{a,1}\big)\mathscr{Y}\\
&= \Theta^a(\mathscr{Y}) t^P_a - \iota_{[t^P_a,\mathscr{Y}]} d\Theta^a + \big([t_a,t_b]^P\mathscr{Y}^b - \lambda_{\mathsf{ad}}(t_a,\mathscr{Y})\big)\Theta^a\\
&= \mathscr{Y} - \Theta^b([t^P_a,\mathscr{Y}])\iota_{t^P_b} d\Theta^a + \Theta^b\big([[t_a,t_b]^P,\mathscr{Y}]\big)\Theta^a\\
&= \mathscr{Y} + \Theta^b([t^P_a,\mathscr{Y}]) t^a([t_b,t_c])\Theta^c + t^c([t_a,t_b])\Theta^b([t^P_c,\mathscr{Y}])\Theta^a\\
&= \mathscr{Y}
\end{aligned}
$$

using Lemma 4.11, Lemma 2.7 and the computation of $\{A^P,\mathscr{Y}\}$ in §4.15 (with $\lambda = -\lambda_{\mathsf{ad}}$ but without the assumption that $\mathscr{Y}$ is G-invariant).

Now consider the null-homotopic operator on the Feigin complex (6.4) that is defined by the first expression below but is also equivalent to the other expressions by Lemma 6.3:

$$
e = \big[Q_0,\, \gamma_0 Q_0 \gamma_0\big] = \big[Q_0,\gamma_0\big]^2 = \Big(L^{\Pi\mathfrak{g}}_0\otimes 1 + 1\otimes (t^P_{a,-1}\Theta^a)_0\Big)^2.
$$

Let e_1 denote its restriction to weight one. By the calculations above, we have

$$
\begin{aligned}
&e_1(u_1\otimes f) = u_1\otimes f, && u_1\in\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{g})_1,\ f\in C^\infty(P)\\
&e_1(u_0\otimes\alpha) = u_0\otimes\alpha_v, && u_0\in\mathcal{O}(\Pi\mathfrak{g}),\ \alpha\in\Omega^1(P)\\
&e_1(u_0\otimes\mathscr{X}) = u_0\otimes(\tau_i\mathscr{X}^j)\rho(\Theta)_{ij}, && u_0\in\mathcal{O}(\Pi\mathfrak{g}),\ \mathscr{X}\in\mathcal{T}_h(P)\\
&e_1(u_0\otimes\mathscr{Y}) = u_0\otimes\mathscr{Y}, && u_0\in\mathcal{O}(\Pi\mathfrak{g}),\ \mathscr{Y}\in\mathcal{T}_v(P)
\end{aligned}
$$

Notice that e_1 is an idempotent. Therefore the image of $1-e_1$ is a quasi-isomorphic subcomplex. This is the subcomplex stated in the lemma. ☐

§ 6.5. The component of weight one. By Lemma 6.4, the weight-one component of (6.1) is

$$
\begin{aligned}
\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})_1 &\cong H^0\big(\mathfrak{g},\mathcal{D}^{\mathrm{ch}}_h(P)_1\big) = \mathcal{D}^{\mathrm{ch}}_h(P)^G_1\\
&= \Omega^1_h(P)^G \oplus \big\{\mathscr{X} - (\tau_i\mathscr{X}^j)\rho(\Theta)_{ij} : \mathscr{X}\in\mathcal{T}_h(P)^G\big\}\\
&= \Omega^1(M)\oplus\big\{\widetilde{X} - (\tau_i\widetilde{X}^j)\rho(\Theta)_{ij} : X\in\mathcal{T}(M)\big\}
\end{aligned}
\tag{6.6}
$$

[9] Keep in mind that we identify $\Omega^*(M)$ with the basic subspace of $\Omega^*(P)$, and denote the horizontal lift of any $X\in\mathcal{T}(M)$ by $\widetilde{X}\in\mathcal{T}_h(P)^G$. Notice that by (4.7) the G-invariance of $\widetilde{X} = \widetilde{X}^i\tau_i$ means

$$
A^P\widetilde{X}^i = -\rho(A)_{ij}\widetilde{X}^j \quad \text{for } A\in\mathfrak{g}. \tag{6.7}
$$

[9] By Lemma 5.6, there is a map of vertex algebras $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\hat{\mathfrak{g}}}\to\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$. Comparing (4.16) and (6.6), we see that the weight-one component of the said map is surjective and its kernel is $\mathcal{T}_v(P)^G$ (since $\lambda = -\lambda_{\mathsf{ad}}$).

Also notice that the operator $\nabla X \in \Gamma(\operatorname{End} TM)$ lifts horizontally to

$$\widetilde{\nabla X} = \big(d\widetilde{X}^j + \rho(\Theta)_{jk}\widetilde{X}^k\big) \otimes \tau_j = (\tau_i \widetilde{X}^j)\, \tau^i \otimes \tau_j \tag{6.8}$$

where the first equality simply expresses the relation between ∇ and Θ, and the second equality follows from (6.7). The unusual-looking term in (6.6) can be given a global interpretation using (6.8).

§ 6.6. The associated vertex algebroid. In view of (6.5) and (6.6), the extended Lie algebroid associated to $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ is $(C^\infty(M), \Omega^1(M), \mathcal{T}(M))$ with the usual structure maps (see §A.5). More precisely, we are identifying each $X \in \mathcal{T}(M)$ with

$$\ell X := \text{ class of } \widetilde{X} - (\tau_i \widetilde{X}^j)\rho(\Theta)_{ij} \quad \in \Gamma^{\mathrm{ch}}(\pi,\mathbb{C})_1.$$

[10] Consider the vertex algebroid associated to $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ (see §A.5 again):

$$\big(C^\infty(M), \Omega^1(M), \mathcal{T}(M), \bullet, \{\ \}, \{\ \}_\Omega\big).$$

The explicit expressions of $\bullet, \{\ \}, \{\ \}_\Omega$ may not seem very meaningful, but we will obtain them below for the sole purpose of recovering the corresponding known expressions for $\mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$. Let V^f be the vertex algebra freely generated by the above vertex algebroid, and

$$\Phi : V^f \to \Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$$

the resulting universal map (see §A.7). By construction, Φ is an isomorphism in the two lowest weights. In fact, we will see that Φ is an isomorphism in all weights, which means $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ is freely generated by its associated vertex algebroid.

Proposition 6.7. *For $f \in C^\infty(M)$ and $X, Y \in \mathcal{T}(M)$, we have*

$$\begin{aligned}
\ell X \bullet f \quad &= (\nabla X) f \\
\{\ell X, \ell Y\} \quad &= -\operatorname{Tr}(\nabla X \cdot \nabla Y) \\
\{\ell X, \ell Y\}_\Omega &= \operatorname{Tr}\Big(-\nabla(\nabla X)\cdot \nabla Y + \nabla X \cdot \iota_Y R - \iota_X R \cdot \nabla Y\Big) + \tfrac{1}{2}\iota_X \iota_Y H
\end{aligned}$$

Proof. Let us verify these expressions at the level of $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$. Recall the normal-ordered expansions and commutation relations in Corollary 4.12, as well as those in the larger collection (A.3)–(A.4). By definition (A.2), $\ell X \bullet f$ is represented in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ by

$$\begin{aligned}
&\big(\widetilde{X} - \tau_i \widetilde{X}^j \rho(\Theta)_{ij}\big)_{-1} f - \big(\widetilde{fX} - \tau_i(\widetilde{fX})^j \rho(\Theta)_{ij}\big) \\
&\qquad = \big(\tau_{i,-1}\widetilde{X}^i_0 + \widetilde{X}^i_{-1}\tau_{i,0}\big) f - f\widetilde{X} + \big(\tau_i(f\widetilde{X}^j) - f\tau_i\widetilde{X}^j\big)\rho(\Theta)_{ij} \\
&\qquad = f\widetilde{X} + (\tau_i f) d\widetilde{X}^i - f\widetilde{X} + (\tau_i f)\widetilde{X}^j \rho(\Theta)_{ij} \\
&\qquad = (\tau_i f)\big(d\widetilde{X}^i + \rho(\Theta)_{ij}\widetilde{X}^j\big)
\end{aligned}$$

which by (6.8) proves the claim. Now keep (6.7) in mind. By (A.2), $\{\ell X, \ell Y\}$ is represented by

$$\begin{aligned}
&\big(\widetilde{X} - \tau_i \widetilde{X}^j \rho(\Theta)_{ij}\big)_1 \big(\widetilde{Y} - \tau_k \widetilde{Y}^\ell \rho(\Theta)_{k\ell}\big) \\
&\qquad = \big(\widetilde{X}^i_1 \tau_{i,0} + \widetilde{X}^i_0 \tau_{i,1}\big)\tau_{k,-1}\widetilde{Y}^k \\
&\qquad = -[\tau_{k,-1}, \widetilde{X}^i_1]\, \tau_{i,0}\widetilde{Y}^k - [[\tau_{i,0}, \tau_{k,-1}], \widetilde{X}^i_1]\, \widetilde{Y}^k + \widetilde{X}^i_0\, [\tau_{i,1}, \tau_{k,-1}]\, \widetilde{Y}^k \\
&\qquad = -(\tau_k \widetilde{X}^i)(\tau_i \widetilde{Y}^k) + \widetilde{Y}^k \Omega(\tau_i, \tau_k)^P \widetilde{X}^i - \widetilde{X}^i \Omega(\tau_i, \tau_k)^P \widetilde{Y}^k + 2\mathrm{Ric}_{ik}\widetilde{X}^i \widetilde{Y}^k \\
&\qquad = -(\tau_k \widetilde{X}^i)(\tau_i \widetilde{Y}^k)
\end{aligned}$$

[10] For example, it follows from (4.1) that $[\ell X, \ell Y] = \ell[X,Y]$, i.e. the Lie bracket in the said extended Lie algebroid indeed agrees with the usual Lie bracket on $\mathcal{T}(M)$.

which by (6.8) again proves the claim. The computation of $\{\ell X, \ell Y\}_\Omega$ is more tedious and will only be sketched here. First of all, by (A.2) and Lemma 2.7, $\{\ell X, \ell Y\}_\Omega$ is represented in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ by

$$\begin{aligned}
&\big(\widetilde{X} - \tau_i \widetilde{X}^j\,\rho(\Theta)_{ij}\big)_0\big(\widetilde{Y} - \tau_k \widetilde{Y}^\ell\,\rho(\Theta)_{k\ell}\big) - \big(\widetilde{[X,Y]} - \tau_i \widetilde{[X,Y]}^j \rho(\Theta)_{ij}\big)\\
&= \big(\tau_{i,-1}\widetilde{X}^i_1 + \widetilde{X}^i_0\tau_{i,0} + \widetilde{X}^i_{-1}\tau_{i,1}\big)\tau_{k,-1}\widetilde{Y}^k - \widetilde{[X,Y]}\\
&\qquad + \iota_{\widetilde{Y}} d\big(\tau_i \widetilde{X}^j\,\rho(\Theta)_{ij}\big) - L_{\widetilde{X}}\big(\tau_i \widetilde{Y}^j\,\rho(\Theta)_{ij}\big) + \tau_i \widetilde{[X,Y]}^j \rho(\Theta)_{ij}
\end{aligned} \tag{6.9}$$

With some work, it turns out the first line in (6.9) can be rewritten as

$$-\Omega(\widetilde{X},\widetilde{Y})^P - \tau_i \widetilde{Y}^k\, d(\tau_k \widetilde{X}^i) + \tfrac{1}{2}\iota_{\widetilde{X}}\iota_{\widetilde{Y}} H + \lambda^*(\Omega(\widetilde{X},\widetilde{Y}),\Theta)$$

while the second line in (6.9), thanks to the first Bianchi identity, equals

$$\big((\tau_i \widetilde{X}^k)(\tau_k \widetilde{Y}^j) - (\tau_i \widetilde{Y}^k)(\tau_k \widetilde{X}^j)\big)\rho(\Theta)_{ij} + \tau_i \widetilde{X}^j\,\iota_{\widetilde{Y}}\rho(\Omega)_{ij} - \tau_i \widetilde{Y}^j\,\iota_{\widetilde{X}}\rho(\Omega)_{ij} - \lambda_\rho(\Omega(\widetilde{X},\widetilde{Y}),\Theta).$$

Notice that $\lambda^* = \lambda_\rho$ in our current setting (see §6.1) and by the proof of Lemma 6.4 we may ignore the term $-\Omega(\widetilde{X},\widetilde{Y})^P$. Then it follows from (4.7) and (6.7) that $\{\ell X, \ell Y\}_\Omega$ is also represented by

$$-(\tau_\ell \tau_k \widetilde{X}^i)(\tau_i \widetilde{Y}^k)\tau^\ell + \tau_i \widetilde{X}^j\,\iota_{\widetilde{Y}}\rho(\Omega)_{ij} - \tau_i \widetilde{Y}^j\,\iota_{\widetilde{X}}\rho(\Omega)_{ij} + \tfrac{1}{2}\iota_{\widetilde{X}}\iota_{\widetilde{Y}} H.$$

By (6.8) again this proves our last claim. □

Remark. This result recovers the vertex algebroid described in Theorem 2.5a (as ∇ is now torsion-free). In other words, we have $V^f = \mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$.

§ 6.8. The conformal vector. According to §6.1 and Lemma 6.3, the vertex algebra $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ has a conformal vector ν^M of central charge $2d = 2\dim M$, represented by

$$\begin{aligned}
\nu^{\Pi\mathfrak{g}} \otimes \mathbf{1} + \mathbf{1} \otimes \nu^P - Q_0\gamma &= \mathbf{1} \otimes \left(\nu^P - t^P_{a,-1}\Theta^a\right)\\
&= \mathbf{1} \otimes \left(\tau_{i,-1}\tau^i - \tfrac{1}{2}\lambda_\rho(\Theta_{-1}\Theta)\right)
\end{aligned} \tag{6.10}$$

The key to understanding the whole structure of $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ is the observation that ν^M is generated by the associated vertex algebroid (in a certain way). In fact, as we will see, its representative (6.10) can be expressed entirely in terms of elements of (6.6).

For the following statement, notice that by assumption $\pi : P \to M$ is a lifting of the special orthogonal frame bundle $F_{SO}(TM) \to M$ (see §6.1).

Proposition 6.9. *The vertex algebra V^f has a conformal vector ν^f with $\Phi(\nu^f) = \nu^M$ (see §6.6). Given an open subset $U \subset M$ and a smooth section $\sigma : U \to \pi^{-1}(U) \subset P$ of π, there is a local expression*

$$\nu^f|_U = (\ell\sigma_i)_{-1}\sigma^i + \tfrac{1}{2}\mathrm{Tr}\left(\Gamma^\sigma_{-1}\Gamma^\sigma\right) + \sigma^i([\sigma_j,\sigma_k])\,\sigma^k_{-1}\Gamma^\sigma_{ji}$$

[11] *where $(\sigma_1,\dots,\sigma_d)$ is the $C^\infty(U)$-basis of $\mathcal{T}(U)$ induced by σ; $(\sigma^1,\dots,\sigma^d)$ the dual basis of $\Omega^1(U)$; and $\Gamma^\sigma = \rho(\sigma^*\Theta) \in \Omega^1(U) \otimes \mathfrak{so}_d$. In particular, $\nu^f \in \mathcal{F}_{\preceq(-1;-1)}V^f$ (see §A.11).*

Proof. Consider the smooth map $g : \pi^{-1}(U) \to G$ defined by $\sigma(\pi(p)) = p \cdot g(p)$ for $p \in \pi^{-1}(U)$. To ease notations, we also write $\rho(g) : \pi^{-1}(U) \to SO(\mathbb{R}^d)$ simply as g. Then we have

$$\widetilde{\sigma}_i = g_{ri}\tau_r, \qquad \sigma^i = g_{ri}\tau^r, \qquad \tau_r = g_{ri}\widetilde{\sigma}_i, \qquad \tau^r = g_{ri}\sigma^i. \tag{6.11}$$

[11] Recall from §2 that, like most constructions in this paper, V^f is the space of global sections of an underlying sheaf.

Since $\nabla\sigma_i = \Gamma^\sigma_{ji} \otimes \sigma_j$, it follows from (6.8) and (6.11) that

$$dg + \rho(\Theta) \cdot g = (\tau_k g)\tau^k = g \cdot \Gamma^\sigma \tag{6.12}$$

where $\cdot$ denotes matrix multiplication.

The main task here is to express (6.10) entirely in terms of σ. Let us first write down

$$\begin{aligned}\tau_{r,-1}\tau^r &= \tau_{r,-1}g_{ri,0}\sigma^i = (\tau_{r,-1}g_{ri})_{-1}\sigma^i - \big(g_{ri,-1}\tau_{r,0} + g_{ri,-2}\tau_{r,1}\big)g_{si}\tau^s \\ &= \widetilde{\sigma}_{i,-1}\sigma^i - (\tau_r g_{si})\tau^s_{-1}dg_{ri} - \mathrm{Tr}\left(g^{-1}\cdot\rho(\Theta)_{-1}dg\right) - \tfrac{1}{2}\mathrm{Tr}\left(g^{-1}\cdot T dg\right)\end{aligned}$$

using (6.11), Corollary 4.12 and the Lie derivative $L_{\tau_r}\tau^s = \rho(\Theta)_{sr}$ implied by (4.7). Then we work on each term separately, using (6.11) and (6.12) a number of times:

$$\begin{aligned}\text{2nd term} &= (\tau_r g_{si})\tau^s_{-1}\big(\rho(\Theta)_{rt}g_{ti} - g_{rj}\Gamma^\sigma_{ji}\big) \\ &= -g_{si}(\tau_r g_{ti})\tau^s_{-1}\rho(\Theta)_{rt} + g_{si}g_{rj}(\tau_r g_{sk})\sigma^k_{-1}\Gamma^\sigma_{ji} \\ &= -(\tau_r\widetilde{\sigma}^t_i)\sigma^i_{-1}\rho(\Theta)_{rt} + g_{si}g_{rk}(\tau_r g_{sj})\sigma^k_{-1}\Gamma^\sigma_{ji} + \sigma^i([\widetilde{\sigma}_j,\widetilde{\sigma}_k])\sigma^k_{-1}\Gamma^\sigma_{ji} \\ &= -(\tau_r\widetilde{\sigma}^t_i)\sigma^i_{-1}\rho(\Theta)_{rt} + \mathrm{Tr}\left(\Gamma^\sigma_{-1}\Gamma^\sigma\right) + \sigma^i([\sigma_j,\sigma_k])\sigma^k_{-1}\Gamma^\sigma_{ji} \\ \text{3rd term} &= \lambda_\rho(\Theta_{-1}\Theta) - \mathrm{Tr}\left(g^{-1}\cdot\rho(\Theta)_{-1}\cdot g\cdot\Gamma^\sigma\right) \\ \text{4th term} &= -\tfrac{1}{2}T\big(\mathrm{Tr}\,(g^{-1}dg)\big) - \tfrac{1}{2}\mathrm{Tr}\left(g^{-1}\cdot dg_{-1}\cdot g^{-1}\cdot dg\right) \\ &= 0 - \tfrac{1}{2}\lambda_\rho(\Theta_{-1}\Theta) + \mathrm{Tr}\left(g^{-1}\cdot\rho(\Theta)_{-1}\cdot g\cdot\Gamma^\sigma\right) - \tfrac{1}{2}\mathrm{Tr}\left(\Gamma^\sigma_{-1}\Gamma^\sigma\right)\end{aligned}$$

These calculations together yield an identity in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(\pi^{-1}(U))$:

$$\tau_{r,-1}\tau^r - \tfrac{1}{2}\lambda_\rho(\Theta_{-1}\Theta) = \big(\widetilde{\sigma}_i - \tau_r\widetilde{\sigma}^t_i\,\rho(\Theta)_{rt}\big)_{-1}\sigma^i + \tfrac{1}{2}\mathrm{Tr}\left(\Gamma^\sigma_{-1}\Gamma^\sigma\right) + \sigma^i([\sigma_j,\sigma_k])\sigma^k_{-1}\Gamma^\sigma_{ji}. \tag{6.13}$$

The left hand side, as observed in §6.8, represents $\nu^M \in \Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$; and it is defined globally on P. Now that the right hand side is manifestly generated by the subspace (6.6), it defines an element $\nu^f \in V^f$ such that $\Phi(\nu^f) = \nu^M$.

It remains to show that ν^f is a conformal vector of V^f. According to [Kac98, FB04], this amounts to checking: (i) $\nu^f_{-1} = T$, (ii) $\nu^f_0 = L_0$, and (iii) $\nu^f_2\nu^f \in \mathbb{C}$. For (i) and (ii) it suffices to check them on

$$C^\infty(U) \cup \{\ell\sigma_1,\ldots,\ell\sigma_d\}$$

because V^f is locally generated by these elements. In fact, since $\Phi(\nu^f) = \nu^M$ is known to be conformal and Φ is an isomorphism in weights 0 and 1, everything we need to check automatically holds except for the equation $\nu^f_{-1}\ell\sigma_i = T(\ell\sigma_i)$ in weight 2. This equation can be verified by a straightforward calculation, which we omit. □

Remark. Since we have already observed that $V^f = \mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$, this result recovers the conformal vector ν^0 described in Theorem 2.5b (as Γ^σ is now traceless).

Corollary 6.10. *The map of vertex algebras $\Phi : V^f \to \Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ (see §6.6) is an isomorphism.*

Proof. By Proposition 6.9, the conformal vector $\nu^M \in \Gamma^{\mathrm{ch}}(\pi,\mathbb{C})$ belongs to $\mathcal{F}_{\preceq(-1;-1)}$. Then Lemma A.12 applies so that Φ is surjective. By construction, the ideal $\ker\Phi \subset V^f$ is trivial in weights 0 and 1. Then by Proposition 6.9 again and Lemma A.13, $\ker\Phi$ is in fact trivial in all weights. □

Propositions 6.7, 6.9 and Corollary 6.10 together show that $\Gamma^{\mathrm{ch}}(\pi,\mathbb{C}) \cong \mathcal{D}^{\mathrm{ch}}_{\nabla,H}(M)$ as conformal vertex algebras. This fulfills the goal of this section. Let us summarize our work in the form of a new description of algebras of CDOs.

Theorem 6.11. *Suppose $\pi : P \to M$ is a smooth prinicpal G-bundle and $\rho : G \to SO(\mathbb{R}^d)$ is a representation such that there is an isomorphism $P \times_\rho \mathbb{R}^d \cong TM$. Given a principal frame $(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, G)$-algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ (see Theorem 4.14), the zeroth semi-infinite cohomology*

$$\Gamma^{\mathrm{ch}}(\pi, \mathbb{C}) = H^{\frac{\infty}{2}+0}\big(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\big)$$

is an algebra of CDOs on M. Up to isomorphism, every algebra of CDOs on M arises this way. $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ as a vertex algebra is freely generated by its weight-zero and weight-one components, which are represented by the following subspaces of $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ bijectively:

$$\pi^* C^\infty(M), \qquad \pi^*\Omega^1(M) \oplus \big\{\widetilde{X} - \tau_i \widetilde{X}^j \,\rho(\Theta)_{ij} : X \in \mathcal{T}(M)\big\}.$$

(For details of the latter representation, see the proof of Lemma 6.4.) Moreover, $\Gamma^{\mathrm{ch}}(\pi, \mathbb{C})$ has a conformal vector of central charge $2d = 2\dim M$, represented in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ by

$$\tau_{i,-1}\tau^i - \tfrac{1}{2}\lambda_\rho(\Theta_{-1}\Theta). \qquad \square$$

To conclude this section, let us sketch an extension of Theorem 6.11.

§ 6.12. Generalization to supermanifolds. Suppose $\pi : P \to M$ and $\rho : G \to SO(\mathbb{R}^d)$ are the same as above; also let $\rho' : G \to U(\mathbb{C}^r)$ be another representation, $E = P \times_{\rho'} \mathbb{C}^r$ the associated vector bundle and ΠE the corresponding cs-manifold. [DM99] The G-action on $\mathcal{O}(\Pi\mathbb{R}^r) \otimes \mathbb{C} = \wedge^*(\mathbb{C}^r)^\vee$ induced by ρ' lifts to an inner $(\hat{\mathfrak{g}}_{\lambda_{\rho'}}, G)$-action on $\mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^r)$, a fermionic analogue of §2.1 (see also Definition 3.4). By Theorem 4.14, there exists a principal frame $(\hat{\mathfrak{g}}_\lambda, G)$-algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ with $\lambda + \lambda_{\rho'} = -\lambda_{\mathsf{ad}}$ if and only if

$$\lambda^*(P) = (\lambda_\rho - \lambda_{\rho'})(P) = 0 \qquad \Longleftrightarrow \qquad p_1(M) - ch_2(E) = 0.$$

In this case, we can apply Definition 5.5 to construct a vertex superalgebra

$$\Gamma^{\mathrm{ch}}(\pi, \mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^r)) = H^{\frac{\infty}{2}+0}\Big(\hat{\mathfrak{g}}_{-\lambda_{\mathsf{ad}}}\,,\, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes \mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^r)\Big).$$

Moreover, the $(\hat{\mathfrak{g}}_{\lambda_{\rho'}}, G)$-action on $\mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^r)$ is primary if and only if

$$\rho'(\mathfrak{g}) \subset \mathfrak{su}_r \qquad \Longleftrightarrow \qquad c_1(E) = 0.$$

In this case, $\Gamma^{\mathrm{ch}}(\pi, \mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^r))$ has a conformal vector of central charge $2(d - r)$. By a similar analysis, $\Gamma^{\mathrm{ch}}(\pi, \mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^r))$ is an algebra of CDOs on ΠE in the sense of [Che11]. In particular, in the case $\rho' = \rho_{\mathbb{C}}$ (i.e. $E = TM_{\mathbb{C}}$), both obstructions are trivial and $\Gamma^{\mathrm{ch}}(\pi, \mathcal{D}^{\mathrm{ch}}(\Pi\mathbb{R}^d))$ is the smooth chiral de Rham algebra of M.

§7. Example: Spinor Module over CDOs

Given a spin manifold M with a trivialization of its first Pontrjagin form, we are interested in the space of sections of the hypothetical "spinor bundle on the formal loops of M" as a module over its "differential operators". The object of study in this section *should* have the above geometric meaning, but is more precisely an example of the construction introduced in §5 defined using semi-infinite cohomology. Following a similar strategy as in §6, we analyze this spinor module so as to obtain a more explicit description in terms of generating data and relations. It is hoped that a deeper understanding, including the identification of an appropriate Dirac operator, will lead to a useful geometric theory of the Witten genus. (This was in fact the original motivation of the paper.)

In this section, $G = \mathrm{Spin}_d$ and $\mathfrak{g} = \mathfrak{spin}_d = \mathfrak{so}_d$ where $d = 2d'$ is even (except in §7.1). Since $\mathfrak{so}_d$ is simple, we use the more common notation $(\widehat{\mathfrak{so}}_d)_k$ in place of $(\widehat{\mathfrak{so}}_d)_{k\lambda_0}$ (see §3.1), where λ_0 is the normalized Killing form, i.e. $\lambda_0(A,B) = \frac{1}{2}\mathrm{Tr}\, AB$ for $A, B \in \mathfrak{so}_d$.

§ 7.1. The Ramond Clifford algebra. Let $C\ell$ be the unital $\mathbb{Z}/2\mathbb{Z}$-graded associative $\mathbb{C}$-algebra with the following generators and relations

$$e_{i,n} \text{ odd}, \;\; i = 1, \ldots, d, \;\; n \in \mathbb{Z}, \qquad [e_{i,n}, e_{j,m}] = e_{i,n}e_{j,m} + e_{j,m}e_{i,n} = -2\delta_{ij}\delta_{n+m,0}. \tag{7.1}$$

(The notations are chosen to resemble those in [LM89].) Suppose W is a $C\ell$-module with the property that for each $w \in W$, there exists $N \in \mathbb{Z}$ such that $e_{i,n}w = 0$ for $n > N$. Then the operators

$$A_n^{C\ell} = \frac{1}{4} A_{ji}\, e_{i,n-r} e_{j,r} \quad \text{for } A \in \mathfrak{so}_d, \; n \in \mathbb{Z} \tag{7.2}$$

define an $(\widehat{\mathfrak{so}}_d)_1$-action on W. On the other hand, the operators

$$L_n^{C\ell} = -\frac{1}{8}\sum_{r\geq 0}(2r-n)e_{i,n-r}e_{i,r} + \frac{1}{8}\sum_{r<0}(2r-n)e_{i,r}e_{i,n-r} + \frac{d}{16}\delta_{n,0} \quad \text{for } n \in \mathbb{Z} \tag{7.3}$$

define a Virasoro action on W of central charge $d/2$. This is in fact the Sugawara construction associated to the above $(\widehat{\mathfrak{so}}_d)_1$-action, and accordingly satisfies

$$[L_n^{C\ell}, A_m^{C\ell}] = -mA_{n+m}^{C\ell} \quad \text{for } n, m \in \mathbb{Z}.$$

The eigenvalues of $L_0^{C\ell}$ are called weights (as in previous sections) and each $e_{i,n}$ changes weights by $-n$. For more details, see e.g. [Fuc95]

For the rest of the section, $d = 2d'$ is even.

§ 7.2. The spinor representation of $\widehat{\mathfrak{so}}_{2d'}$. Let $C\ell_0$ (resp. $C\ell_{\geq 0}$) be the subalgebra of $C\ell$ generated by those $e_{i,n}$ with $n = 0$ (resp. $n \geq 0$). The finite-dimensional Clifford algebra $C\ell_0$ has a unique (up to isomorphism) irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representation S_0. Let us also regard S_0 as a $C\ell_{\geq 0}$-module on which $\{e_{i,n}\}_{n>0}$ act trivially. Then define a $C\ell$-module by

$$S = C\ell \otimes_{C\ell_{\geq 0}} S_0.$$

By (7.1), S as a vector space is spanned by elements of the form

$$e_{i_p,n_p} \cdots e_{i_1,n_1} s, \qquad n_1 < 0, \;\; (n_p, i_p) < \cdots < (n_1, i_1), \;\; s \in S_0$$

where the indicated pairs are ordered lexicographically. According to §7.1, S admits a $((\widehat{\mathfrak{so}}_{2d'})_1, \mathrm{Spin}_{2d'})$-action as well as a Virasoro action of central charge d', such that the former is primary. Notice that the element displayed above has weight

$$\frac{d'}{8} + |n_1| + \ldots + |n_p|.$$

For convenience, we will write $S_k \subset S$ for the component of weight $(d'/8) + k$.

Given a representation $\rho : \mathfrak{so}_{2d'} \to \mathfrak{gl}(V)$, let λ_ρ denote the invariant symmetric bilinear form on $\mathfrak{so}_{2d'}$ defined by $\lambda_\rho(A,B) = \operatorname{Tr}\rho(A)\rho(B)$. In particular, notice that $\lambda_\rho = 2\lambda_0$ for the standard representation ρ, and $\lambda_{\mathsf{ad}} = (4d'-4)\lambda_0$.

§ 7.3. The spinor module over CDOs. Consider the special case of Definition 5.5 associated to a principal frame $((\hat{\mathfrak{so}}_{2d'})_{3-4d'}, \mathrm{Spin}_{2d'})$-algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ and the $((\hat{\mathfrak{so}}_{2d'})_1, \mathrm{Spin}_{2d'})$-module S:

$$\begin{aligned} \Gamma^{\mathrm{ch}}(\pi, S) &= H^{\frac{\infty}{2}+0}\Big((\hat{\mathfrak{so}}_{2d'})_{4-4d'}\,,\, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\otimes S\Big) \\ &= H^0\Big(\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{so}_{2d'})\otimes\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\otimes S\,,\, Q_0\Big) \end{aligned} \tag{7.4}$$

By Lemma 5.6 and §7.2, $\Gamma^{\mathrm{ch}}(\pi, S)$ is a module over the vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\hat{\mathfrak{so}}_{2d'}}$ and admits a Virasoro action of central charge $5d' = \frac{5}{2}\dim M$. For convenience, let us elaborate on all the data involved and recall some relevant notations.

- Let $(t_1, t_2, \ldots)$ be a basis of $\mathfrak{so}_{2d'}$, $(t^1, t^2, \ldots)$ the dual basis of $(\mathfrak{so}_{2d'})^\vee$, $(\phi^1, \phi^2, \ldots)$ the corresponding coordinates of the supermanifold $\Pi\mathfrak{so}_{2d'}$, and $(\partial_1, \partial_2, \ldots)$ their coordinate vector fields.
- For comments on the conformal vertex superalgebra $\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{so}_{2d'})$, see §6.1.
- For the detailed definition of the vertex algebra $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$, see Theorem 4.14 with $\lambda = (3-4d')\lambda_0$ in mind. Let us mention that it is defined using the following data: a principal $\mathrm{Spin}_{2d'}$-frame bundle $\pi : P \to M$; the Levi-Civita connection $\Theta = \Theta^a \otimes t_a$ on π; and a basic 3-form H on P that satisfies $dH = \lambda_0(\Omega\wedge\Omega)$, where $\Omega = d\Theta + \frac{1}{2}[\Theta\wedge\Theta]$. Also, there is a conformal vector
$$\nu^P = t^P_{a,-1}\Theta^a + \tau_{i,-1}\tau^i - \frac{1}{2}\lambda_0(\Theta_{-1}\Theta)$$
of central charge $2\dim P$. (For the notations see §4.6.)
- For details on the Feigin complex in (7.4), see §5.2. Let us mention that its differential is
$$Q_0 = q_0\otimes 1\otimes 1 + \phi^a_{-r}\otimes t^P_{a,r}\otimes 1 + \phi^a_{-r}\otimes 1\otimes t^{C\ell}_{a,r} \tag{7.5}$$
where q is the odd vector field shown in (5.1). Also the Virasoro operators
$$L^{\Pi\mathfrak{so}}_n\otimes 1\otimes 1 + 1\otimes L^P_n\otimes 1 + 1\otimes 1\otimes L^{C\ell}_n, \qquad n\in\mathbb{Z}$$
preserve the gradation and commute with Q_0.

Let us repeat the geometric ingredients in slightly different words: M is a Riemannian manifold with a spin structure and a 3-form H that satisfies $dH = \frac{1}{2}\operatorname{Tr}(R\wedge R)$, where R is the Riemannian curvature. This can be viewed as the de Rham version of a *string structure*. In the sequel, we will describe $\Gamma^{\mathrm{ch}}(\pi, S)$ more explicitly in terms of generating data (i.e. a subspace and some fields) and their relations.

Throughout this section we identify $\Omega^*(M)$ with the basic subspace of $\Omega^*(P)$.

§ 7.4. The component of the lowest weight. Consider the Feigin complex appearing in (7.4):

$$\Big(\mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{so}_{2d'})\otimes\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\otimes S\,,\, Q_0\Big). \tag{7.6}$$

Since its component of weight $d'/8$ is simply the Chevalley-Eilenberg complex (see §5.1)

$$\Big(\mathcal{O}(\Pi\mathfrak{so}_{2d'})\otimes C^\infty(P)\otimes S_0\,,\, Q_0\Big)$$

the component of (7.4) of weight $d'/8$ is

$$\Gamma^{\mathrm{ch}}(\pi, S)_0 = H^0\big(\mathfrak{so}_{2d'}, C^\infty(P)\otimes S_0\big) = (C^\infty(P)\otimes S_0)^{\mathfrak{so}_{2d'}} = S(M) \tag{7.7}$$

i.e. the space of sections of the spinor bundle on M. Understanding the rest of $\Gamma^{\mathrm{ch}}(\pi, S)$ requires more work, just as in §6.

Lemma 7.5. *Let $\gamma = \partial_a \otimes \Theta^a \in \mathcal{D}^{\mathrm{ch}}(\Pi\mathfrak{so}_{2d'})_1 \otimes \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)_1$. For $n \in \mathbb{Z}$ we have*

$$[Q_0,\, \gamma_n \otimes 1] = L_n^{\Pi\mathfrak{so}} \otimes 1 \otimes 1 + 1 \otimes (t^P_{a,-1}\Theta^a)_n \otimes 1 + 1 \otimes \Theta^a_{n-r} \otimes t^{C\ell}_{a,r}\,.$$

Moreover, for $n \in \mathbb{Z}$ and $\mathscr{Y} = \mathscr{Y}^a t^P_a \in \mathcal{T}_v(P)^{\mathfrak{so}_{2d'}} \subset \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)_1$ we also have

$$[Q_0,\, (\gamma_0(1 \otimes \mathscr{Y}))_n \otimes 1] = 1 \otimes \mathscr{Y}_n \otimes 1 + 1 \otimes \mathscr{Y}^a_{n-r} \otimes t^{C\ell}_{a,r}\,.$$

Proof. Recall the differential Q_0 from (7.5). The first anticommutator is computed as follows

$$\begin{aligned}[Q_0,\, \gamma_n \otimes 1] &= [(q \otimes 1 + \phi^a \otimes t^P_a)_0,\, \gamma_n] \otimes 1 + [\phi^a_{-r}, \partial_{b,n-s}] \otimes \Theta^b_s \otimes t^{C\ell}_{a,r} \\ &= L_n^{\Pi\mathfrak{so}} \otimes 1 \otimes 1 + 1 \otimes (t^P_{a,-1}\Theta^a)_n \otimes 1 + 1 \otimes \Theta^a_{n-r} \otimes t^{C\ell}_{a,r}\end{aligned}$$

using Lemma 6.3 and the fermionic analogue of (2.2). On the other hand, since $\gamma_0(1 \otimes \mathscr{Y}) = \partial_b \otimes \mathscr{Y}^b$, the second anticommutator can be written as

$$\begin{aligned}[Q_0, (\partial_b \otimes \mathscr{Y}^b)_n \otimes 1] &= [(q \otimes 1 + \phi^a \otimes t^P_a)_0, (\partial_b \otimes \mathscr{Y}^b)_n] \otimes 1 + [\phi^a_{-r}, \partial_{b,n-s}] \otimes \mathscr{Y}^b_s \otimes t^{C\ell}_{a,r} \\ &= \big((q \otimes 1 + \phi^a \otimes t^P_a)_0(\partial_b \otimes \mathscr{Y}^b)\big)_n \otimes 1 + 1 \otimes \mathscr{Y}^a_{n-r} \otimes t^{C\ell}_{a,r}\end{aligned}$$

using again the analogue of (2.2). It remains to compute the element $(q \otimes 1 + \phi^a \otimes t^P_a)_0(\partial_b \otimes \mathscr{Y}^b)$, which is the sum of the following:

$$\begin{aligned}&(q_0\partial_b) \otimes \mathscr{Y}^b = [q, \partial_b] \otimes \mathscr{Y}^b = -t^c([t_b, t_d])\phi^d\partial_c \otimes \mathscr{Y}^b \\ &(\phi^a_0 \otimes t^P_{a,0})(\partial_b \otimes \mathscr{Y}^b) = \phi^a\partial_b \otimes t^P_a\mathscr{Y}^b = -t^b([t_a, t_c])\phi^a\partial_b \otimes \mathscr{Y}^c \\ &(\phi^a_1 \otimes t^P_{a,-1})(\partial_b \otimes \mathscr{Y}^b) = \delta^a_b \otimes t^P_{a,-1}\mathscr{Y}^b = 1 \otimes \mathscr{Y}\end{aligned}$$

Indeed, the first line is similar to a computation in the proof of Lemma 6.3; the second follows from the $\mathfrak{so}_{2d'}$-invariance of $\mathscr{Y}$; and the last follows from the analogue of (2.2) and Corollary 4.12. ☐

Preparation. Let $C\ell(M) = (C^\infty(P) \otimes C\ell_0)^{\mathfrak{so}_{2d'}}$ which is the same as the algebra of $C^\infty(M)$-linear endomorphisms of $S(M)$. Given $X \in \mathcal{T}(M)$, let us denote its horizontal lift by $\widetilde{X} = \widetilde{X}^i\tau_i \in \mathcal{T}_h(P)^{\mathfrak{so}_{2d'}}$ (as usual) and its Clifford action by $cX = \widetilde{X}^i \otimes e_{i,0} \in C\ell(M)$. Notice that $\{cX : X \in \mathcal{T}(M)\}$ generates $C\ell(M)$ as an algebra. On the other hand, each $\mathscr{Y} = \mathscr{Y}^a t^P_a \in \mathcal{T}_v(P)^{\mathfrak{so}_{2d'}}$ satisfies

$$\mathscr{Y} \otimes 1 + \mathscr{Y}^a \otimes t_a = 0 \quad \text{on} \quad (C^\infty(P) \otimes S_0)^{\mathfrak{so}_{2d'}} = S(M)$$

and hence represents an endomorphism $c\mathscr{Y} = -\mathscr{Y}^a \otimes t_a \in C\ell(M)$. [12]

§ 7.6. Fields of low weights. Recall Lemma 5.4a for the special case (7.4): the $\widehat{\mathfrak{so}}_{2d'}$-invariant fields on $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S$, including the vertex operators of $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\widehat{\mathfrak{so}}_{2d'}}$, induce fields on $\Gamma^{\mathrm{ch}}(\pi, S)$. Let us now describe a collection of such fields (or rather their Fourier modes); we will later see that their actions on the subspace $\Gamma^{\mathrm{ch}}(\pi, S)_0 = S(M)$ generate the entire space $\Gamma^{\mathrm{ch}}(\pi, S)$.

Given $f \in C^\infty(M)$ and $X \in \mathcal{T}(M)$, there are fields on $\Gamma^{\mathrm{ch}}(\pi, S)$ whose Fourier modes are

$$\begin{aligned} f_n &:= \text{the operator induced by } f_n \otimes 1 \\ \ell X_n &:= \text{the operator induced by } (\widetilde{X} - \tau_i\widetilde{X}^j\,\Theta_{ij})_n \otimes 1 \\ cX_n &:= \text{the operator induced by } \widetilde{X}^i_{n-r} \otimes e_{i,r}\end{aligned} \tag{7.8}$$

[12] To be precise, we are regarding $\mathfrak{so}_d$ as a Lie subalgebra of $C\ell_0$ via the standard inclusion $A \mapsto \frac{1}{4}A_{ji}e_{i,0}e_{j,0}$.

Indeed, the first two are well-defined because both f and $\widetilde{X} - \tau_i \widetilde{X}^j\, \Theta_{ij}$ belong to $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\widehat{\mathfrak{so}}_{2d'}}$ according to §4.15; and so is the last one because for $A \in \mathfrak{so}_{2d'}$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned}\left[A^P_m \otimes 1 + 1 \otimes A^{C\ell}_m,\, \widetilde{X}^i_{n-r} \otimes e_{i,r}\right] &= [A^P_m, \widetilde{X}^i_{n-r}] \otimes e_{i,r} + \widetilde{X}^i_{n-r} \otimes [A^{C\ell}_m, e_{i,r}] \\ &= (A^P \widetilde{X}^i)_{m+n-r} \otimes e_{i,r} + \widetilde{X}^i_{n-r} \otimes (A_{ji} e_{j,m+r}) \\ &= -A_{ij} \widetilde{X}^j_{m+n-r} \otimes e_{i,r} + A_{ji} \widetilde{X}^i_{n-r} \otimes e_{j,m+r} \\ &= 0\end{aligned}$$

where we have used (A.3), the $\mathfrak{so}_{2d'}$-invariance of $\widetilde{X}$ and (7.1)–(7.2). These three types of fields have weights 0, 1 and $\frac{1}{2}$ respectively.

It will be convenient to also consider some other fields generated by those in (7.8). Given $\alpha \in \Omega^1(M)$ and $\mathscr{Y} \in \mathcal{T}_v(P)^{\mathfrak{so}_{2d'}}$, there are fields on $\Gamma^{\mathrm{ch}}(\pi, S)$ whose Fourier modes are

$$\begin{aligned}\alpha_n \quad &:= \text{the operator induced by } \alpha_n \otimes 1 \\ c\mathscr{Y}_n \quad &:= \text{the operator induced by } (\mathscr{Y} + \lambda_0(\Theta, \mathscr{Y}))_n \otimes 1\end{aligned} \tag{7.9}$$

These are well-defined because both α and $\mathscr{Y} + \lambda_0(\Theta, \mathscr{Y})$ belong to $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)^{\widehat{\mathfrak{so}}_{2d'}}$ according to §4.15. To see that α_n is determined by (7.8), it suffices to consider $\alpha = f dg$ and notice that

$$(f dg)_n = -\sum_{r \in \mathbb{Z}} r f_{n-r} g_r \quad \text{for } f, g \in C^\infty(M) \tag{7.10}$$

by (A.3)–(A.4). On the other hand, by Lemma 7.5 we may represent $c\mathscr{Y}_n$ alternatively by

$$-\mathscr{Y}^a_{n-r} \otimes t^{C\ell}_{a,r} + \lambda_0(\Theta, \mathscr{Y})_n \otimes 1. \tag{7.11}$$

The next lemma explains how $c\mathscr{Y}_n$ is determined by (7.8).

Lemma 7.7. *For $\mathscr{Y} \in \mathcal{T}_v(P)^{\mathfrak{so}_{2d'}}$, the operator $c\mathscr{Y}_n$ on $\Gamma^{\mathrm{ch}}(\pi, S)$ is a sum of operators of the form*

$$\sum_{r \geq 0} cX_{n-r}\, cY_r - \sum_{r<0} cY_r\, cX_{n-r} - 2\langle X, \nabla Y\rangle_n + \langle X, Y\rangle_n, \quad X, Y \in \mathcal{T}(M) \tag{7.12}$$

where $\langle\ \rangle$ and ∇ denote the Riemannian metric and Levi-Civita connection on TM.

Proof. Let $\mathscr{Y}_{ij} = \mathscr{Y}^a (t_a)_{ij} \in C^\infty(P)$. As explained above, $c\mathscr{Y}_n$ is represented in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S$ by (7.11), which by (7.2) and (A.4) can be written as

$$\tfrac{1}{4}\left(\mathscr{Y}_{ij,n-r} \otimes 1\right)\left(1 \otimes e_{i,r-s} e_{j,s} - 2\Theta_{ij,r} \otimes 1\right).$$

Since $c\mathscr{Y} = \frac{1}{4}\mathscr{Y}_{ij} \otimes e_{i,0} e_{j,0}$ corresponds to a 2-vector field under the isomorphism $C\ell(M) \cong \Gamma(\wedge^* TM)$, (7.11) can be expressed further as a sum of operators of the form

$$\tfrac{1}{2}\left((\widetilde{X}^i \widetilde{Y}^j - \widetilde{Y}^i \widetilde{X}^j)_{n-r} \otimes 1\right)\left(1 \otimes e_{i,r-s} e_{j,s} - 2\Theta_{ij,r} \otimes 1\right), \quad X, Y \in \mathcal{T}(M) \tag{7.13}$$

where the factor of $\frac{1}{2}$ is included for convenience. Let us show that (7.13) represents (7.12), which will then prove the lemma. For the calculations below, keep in mind (A.3)–(A.4).

Consider the first two terms in (7.12) together. By definition (7.8) they are represented by

$$\sum_{r \geq 0} (\widetilde{X}^i_{n-r-s} \otimes e_{i,s})(\widetilde{Y}^j_{r-t} \otimes e_{j,t}) - \sum_{r<0} (\widetilde{Y}^j_{r-t} \otimes e_{j,t})(\widetilde{X}^i_{n-r-s} \otimes e_{i,s}).$$

In view of (7.1), let us split this sum into three parts. The first part consists of terms with $i \neq j$:

$$\sum_r \sum_{i \neq j} \sum_{s,t} \widetilde{X}^i_{n-r-s} \widetilde{Y}^j_{r-t} \otimes e_{i,s} e_{j,t} = \sum_{i \neq j} \sum_{s,t} (\widetilde{X}^i \widetilde{Y}^j)_{n-s-t} \otimes e_{i,s} e_{j,t}\,.$$

The second part consists of terms with $i = j$ and $s + t \neq 0$, which vanishes by symmetry:

$$\sum_r \sum_i \sum_{s+t\neq 0} \widetilde{X}^i_{n-r-s} \widetilde{Y}^i_{r-t} \otimes e_{i,s} e_{i,t} = \sum_i \sum_{s+t\neq 0} (\widetilde{X}^i \widetilde{Y}^i)_{n-s-t} \otimes e_{i,s} e_{i,t} = 0\,.$$

The last part consists of terms with $i = j$ and $s + t = 0$, which is computed as follows:

$$\begin{aligned}
\sum_{r\geq 0} \sum_{i,s} \widetilde{X}^i_{n-r-s} \widetilde{Y}^i_{r+s} \otimes e_{i,s} e_{i,-s} &- \sum_{r<0} \sum_{i,s} \widetilde{X}^i_{n-r-s} \widetilde{Y}^i_{r+s} \otimes e_{i,-s} e_{i,s} \\
&= \sum_{i,u} \widetilde{X}^i_{n-u} \widetilde{Y}^i_u \otimes \Big(\sum_{s\geq -u} e_{i,-s} e_{i,s} - \sum_{s>u} e_{i,-s} e_{i,s} \Big) \\
&= -\sum_{i,u} (2u+1) \widetilde{X}^i_{n-u} \widetilde{Y}^i_u \otimes 1 \\
&= -(\widetilde{X}^i \widetilde{Y}^i)_n \otimes 1 + 2(\widetilde{X}^i d\widetilde{Y}^i)_n \otimes 1 \\
&= -\langle X, Y\rangle_n \otimes 1 + 2\langle X, \nabla Y\rangle_n \otimes 1 - 2\big(\widetilde{X}^i \widetilde{Y}^j \Theta_{ij}\big)_n \otimes 1
\end{aligned}$$

where we have used (6.8) in the last equality. It follows from these calculations and a little reorganization that (7.12) is indeed represented by (7.13) as claimed. □

Remark. Since we have $cX \cdot cY = -\langle X, Y\rangle \otimes 1 + \frac{1}{2}(\widetilde{X}^i \widetilde{Y}^j - \widetilde{Y}^i \widetilde{X}^j) \otimes e_{i,0} e_{j,0}$ in $C\ell(M)$, the proof above shows that if we are to associate a field to $cX \cdot cY$, it (or rather its Fourier modes) should be given by the first three terms in (7.12). Perhaps this defines the correct normal-ordered product of the fields associated to cX and cY, where the term $-2\langle X, \nabla Y\rangle_n$ is analogous to a normal-order constant.

Now we record the relations between the lowest-weight component $\Gamma^{\mathrm{ch}}(\pi, S)_0 = S(M)$ and the fields introduced in §7.6, as well as the relations between the fields themselves.

Proposition 7.8. *For $f \in C^\infty(M)$, $X \in \mathcal{T}(M)$ and $s \in S(M)$, we have*

$$\begin{gathered}
f_n s = cX_n s = \ell X_n s = 0 \quad \text{for } n > 0 \\
f_0 s = fs, \qquad cX_0 s = cX \cdot s, \qquad \ell X_0 s = \nabla_X s
\end{gathered}$$

where $\cdot$ denotes Clifford multiplication and ∇ the Levi-Civita connection.

Proof. The first line is true simply because $S(M) \subset \Gamma^{\mathrm{ch}}(\pi, S)$ has the lowest weight. The second line follows immediately from (7.7) and (7.8). □

Proposition 7.9. *Recall the maps $\bullet$, $\{\ \}$, $\{\ \}_\Omega$ in Proposition 6.7. For $f, g \in C^\infty(M)$, $X, Y \in \mathcal{T}(M)$ and $n, m \in \mathbb{Z}$, we have the normal-ordered expansions*

$$(fg)_n = f_{n-r} g_r, \quad c(fX)_n = f_{n-r}\, cX_r, \quad \ell(fX)_n = \sum_{r\geq 0} f_{n-r}\, \ell X_r + \sum_{r<0} \ell X_r f_{n-r} - (\ell X \bullet f)_n$$

as well as the supercommutation relations

$$\begin{aligned}
&[f_n, g_m] = 0, \qquad [cX_n, f_m] = 0, \qquad [\ell X_n, f_m] = (Xf)_{n+m} \\
&[\ell X_n, cY_m] = c(\nabla_X Y)_{n+m}, \qquad [cX_n, cY_m] = -2\langle X, Y\rangle_{n+m} \\
&[\ell X_n, \ell Y_m] = \ell[X, Y]_{n+m} - c\,\Omega(\widetilde{X}, \widetilde{Y})^P_{n+m} + (\{\ell X, \ell Y\}_\Omega)_{n+m} + n\{\ell X, \ell Y\}_{n+m}
\end{aligned}$$

where $\langle\ \ \rangle$ and ∇ denote the Riemannian metric and Levi-Civita connection on TM. Notice that the canonical isomorphism $\mathcal{T}_v(P)^{\mathfrak{so}_{2d'}} \cong \Gamma(\mathrm{End}\, TM)$ identifies $-\Omega(\widetilde{X}, \widetilde{Y})^P$ with the Riemannian curvature operator $R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Proof. Let us verify the relations on the level of $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)\otimes S$. For the calculations below, keep in mind (A.3)–(A.4) and (7.8). The first two normal-ordered expansions are easy, e.g.

$$(\widetilde{fX})^i_{n-s}\otimes e_{i,s} = f_{n-s-t}\widetilde{X}^i_t\otimes e_{i,s} = (f_{n-r}\otimes 1)(\widetilde{X}^i_{r-s}\otimes e_{i,s}).$$

The remaining normal-ordered expansion follows immediately from the equation

$$\widetilde{fX} - \tau_i(\widetilde{fX})^j\Theta_{ij} = (\widetilde{X} - \tau_i\widetilde{X}^j\,\Theta_{ij})_{-1}f - \ell X\bullet f$$

obtained from the first calculation in the proof of Proposition 6.7.

For the supercommutators, the first three are easy. The next two are computed as follows, using (6.8) and (7.1) respectively:

$$\begin{aligned}
&\big[(\widetilde{X} - \tau_i\widetilde{X}^j\Theta_{ij})_n\otimes 1,\, \widetilde{Y}^k_{m-r}\otimes e_{k,r}\big]\\
&\qquad = \big[\widetilde{X}_n, \widetilde{Y}^k_{m-r}\big]\otimes e_{k,r} = (\widetilde{X}^i\tau_i\widetilde{Y}^k)_{n+m-r}\otimes e_{k,r} = (\widetilde{\nabla_X Y})^k_{n+m-r}\otimes e_{k,r}\\
&\big[\widetilde{X}^i_{n-r}\otimes e_{i,r},\, \widetilde{Y}^j_{m-s}\otimes e_{j,s}\big]\\
&\qquad = \widetilde{X}^i_{n-r}\widetilde{Y}^j_{m-s}\otimes[e_{i,r},e_{j,s}] = -2\widetilde{X}^i_{n-r}\widetilde{Y}^i_{m+r}\otimes 1 = -2\langle X,Y\rangle_{n+m}\otimes 1
\end{aligned}$$

The last (super)commutator follows immediately from the equations

$$\begin{aligned}
&\big(\widetilde{X} - \tau_i\widetilde{X}^j\,\Theta_{ij}\big)_1\big(\widetilde{Y} - \tau_k\widetilde{Y}^\ell\,\Theta_{k\ell}\big) = \{\ell X, \ell Y\}\\
&\big(\widetilde{X} - \tau_i\widetilde{X}^j\,\Theta_{ij}\big)_0\big(\widetilde{Y} - \tau_k\widetilde{Y}^\ell\,\Theta_{k\ell}\big)\\
&\qquad = \big(\widetilde{[X,Y]} - \tau_i\widetilde{[X,Y]}^j\Theta_{ij}\big) - \Omega(\widetilde{X},\widetilde{Y})^P - \lambda_0(\Omega(\widetilde{X},\widetilde{Y}),\Theta) + \{\ell X,\ell Y\}_\Omega
\end{aligned}$$

together with (7.9); these equations are the results of the second and third calculations in the proof of Proposition 6.7, applied to our current setting where $\lambda^* = \lambda_0$ and $\lambda_\rho = 2\lambda_0$ (see §7.3). ☐

The following construction is manufactured using precisely the information about $\Gamma^{\mathrm{ch}}(\pi,S)$ we have gathered so far: the subspace $S(M)$, the fields in §7.6 and their relations in Propositions 7.8 and 7.9.

§ 7.10. Comparing with a generators-and-relations construction. Let $\mathcal{U}$ be a unital $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra with generators

$$\text{even}:\quad f_n,\ \ell X_n,\ \alpha_n,\ c\mathscr{Y}_n;\qquad\qquad \text{odd}:\quad cX_n$$

for $f\in C^\infty(M)$, $X\in\mathcal{T}(M)$, $\alpha\in\Omega^1(M)$, $\mathscr{Y}\in\mathcal{T}_v(P)^{\mathfrak{so}_{2d'}}$ and $n\in\mathbb{Z}$, such that (i) $1_n=\delta_{n,0}$, (ii) $f\mapsto f_n$, $X\mapsto \ell X_n$, $\cdots$ are linear and (iii) they satisfy the supercommutation relations in Proposition 7.9. The subalgebra $\mathcal{U}_+\subset\mathcal{U}$ generated by $\{f_n,\,\ell X_n,\,cX_n\}_{n\geq 0}$ has an action on $S(M)$ as described in Proposition 7.8. Let $\widetilde{W}^f = \mathcal{U}\otimes_{\mathcal{U}_+} S(M)$ and $W^f = \widetilde{W}^f/\sim$ be the quotient obtained by imposing the normal-ordered expansions in (7.10), Lemma 7.7 and Proposition 7.9. Define an operator L_0^f on W^f by

$$L^f_0|_{S(M)} = \frac{d'}{8},\qquad [L^f_0, f_n] = -nf_n,\qquad [L^f_0,\ell X_n] = -n\,\ell X_n,\qquad [L^f_0, cX_n] = -n\,cX_n \tag{7.14}$$

which are consistent with all the above relations; its eigenvalues are called weights.

By construction, there is a unique linear map

$$\Phi: W^f\to\Gamma^{\mathrm{ch}}(\pi,S)$$

that restricts to the identity on $S(M)$ [13] and respects all the above operators. Clearly Φ is weight-preserving. As we will see later, Φ is in fact an isomorphism and thus provides a generators-and-relations description of $\Gamma^{\mathrm{ch}}(\pi,S)$.

[13] Since the composition $\widetilde{W}^f\twoheadrightarrow W^f\to\Gamma^{\mathrm{ch}}(\pi,S)$ restricts to the identity on $1\otimes S(M)\subset\widetilde{W}^f$, the quotient map must be injective there. Hence we may indeed identify $S(M)$ as a subspace of W^f.

Remark. The relations imposed in the construction of W^f are not subject to any higher relations. For example, given $X, Y \in \mathcal{T}(M)$ and $s \in S(M)$, the expression $[\ell X_0, cY_0]s$ can be evaluated by either taking the commutator first or applying the action on $S(M)$ first, but the results are identical automatically due to the fact that the Levi-Civita connection is compatible with Clifford multiplication. [LM89] Similarly, other composites of relations (e.g. Jacobi identities) are already implied by the geometric data.

§ 7.11. The Virasoro action. According to §7.3 and Lemma 7.5, the following operators

$$(\tau_{i,-1}\tau^i)_n \otimes 1 + 1 \otimes L_n^{C\ell} - \Theta^a_{n-r} \otimes t^{C\ell}_{a,r} - \tfrac{1}{2}\lambda_0(\Theta_{-1}\Theta)_n \otimes 1, \qquad n \in \mathbb{Z} \tag{7.15}$$

on $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S$ induce a Virasoro action on $\Gamma^{\mathrm{ch}}(\pi, S)$ of central charge $5d'$; denote the induced operators by L_n^M. The key to understanding the entire structure of $\Gamma^{\mathrm{ch}}(\pi, S)$ is the observation that L_0^M is generated by the fields described in §7.6 (in a certain way).

For the following statement, recall that $\pi : P \to M$ is by assumption a lifting of the orthogonal frame bundle of TM (see §7.3).

Proposition 7.12. *Let $U \subset M$ be an open subset and $\sigma : U \to \pi^{-1}(U) \subset P$ a smooth section of π. Both weight operators L_0^f on W^f (see §7.10) and L_0^M on $\Gamma^{\mathrm{ch}}(\pi, S)$ have the local expression:* [14]

$$\begin{aligned}\sum_{r\geq 0}\sigma^i_{-r}(\ell\sigma_i)_r + \sum_{r<0}(\ell\sigma_i)_r\sigma^i_{-r} + \frac{3}{4}\operatorname{Tr}(\Gamma^\sigma_{-r}\Gamma^\sigma_r) + \sigma^i([\sigma_j,\sigma_k])_{-r}\,\sigma^k_{-s}(\Gamma^\sigma_{ji})_{r+s} \\ - \frac{1}{2}\sum_{r>0} r(c\sigma_i)_{-r}(c\sigma_i)_r + \frac{1}{4}(c\sigma_i)_{-r}(c\sigma_j)_{-s}(\Gamma^\sigma_{ij})_{r+s} + \frac{d'}{8}\end{aligned} \tag{7.16}$$

where $(\sigma_1, \dots, \sigma_{2d'})$ is the $C^\infty(U)$-basis of $\mathcal{T}(U)$ induced by σ; $(\sigma^1, \dots, \sigma^{2d'})$ the dual basis of $\Omega^1(U)$; and $\Gamma^\sigma = \sigma^*\Theta \in \Omega^1(U) \otimes \mathfrak{so}_{2d'}$.

Proof. First consider L_0^M. As explained above, L_0^M is represented in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S$ by $(7.15)_{n=0}$, which we would like to split into two parts:

$$\ell' = \left(\tau_{i,-1}\tau^i - \tfrac{1}{2}\operatorname{Tr}(\Theta_{-1}\Theta)\right)_0 \otimes 1, \qquad \ell'' = 1 \otimes L_0^{C\ell} - \Theta^a_{-r} \otimes t^{C\ell}_{a,r} + \tfrac{1}{4}\operatorname{Tr}(\Theta_{-1}\Theta)_0 \otimes 1.$$

Since the calculation in the proof of Proposition 6.9 is valid for any principal frame algebra, ℓ' can be expressed as the zeroth mode of (6.13). Our main task is to express ℓ'' entirely in terms of σ as well.

For the calculations below, keep in mind (A.3)–(A.4) and adopt again the notations in the proof of Proposition 6.9. Let $\theta = g^{-1}dg$. Also let us introduce a (slight abuse of) notation:

$$(c\sigma_i)_r = \tilde{\sigma}^j_{i,r-s} \otimes e_{j,s} = g_{ji,r-s} \otimes e_{j,s}, \qquad i = 1, \dots, 2d', \;\; r \in \mathbb{Z}. \tag{7.17}$$

By (7.1) these operators satisfy the anticommutation relations

$$[(c\sigma_i)_r, (c\sigma_j)_s] = -2\delta_{ij}\delta_{r+s,0}. \tag{7.18}$$

Now the first term of ℓ'' can be written as the first line below by (7.3) and (7.17), and the subsequent rearrangement is valid because every sum in sight is finite when applied to an arbitrary element:

$$\begin{aligned}1 \otimes L_0^{C\ell} &= \frac{1}{4}\Bigg(-\sum_{r\geq 0} r\,(c\sigma_j)_s(c\sigma_k)_t + \sum_{r<0} r\,(c\sigma_k)_t(c\sigma_j)_s\Bigg)\big(g_{ij,-r-s}g_{ik,r-t} \otimes 1\big) + \frac{d'}{8} \\ &= \frac{1}{4}\Bigg(-\sum_{t\geq 0} r\,(c\sigma_j)_s(c\sigma_k)_t + \sum_{t<0} r\,(c\sigma_k)_t(c\sigma_j)_s - \sum_{r\geq 0>t} r\,[(c\sigma_j)_s, (c\sigma_k)_t] \\ &\qquad + \sum_{r<0\leq t} r\,[(c\sigma_j)_s, (c\sigma_k)_t]\Bigg)\big(g_{ij,-r-s}g_{ik,r-t} \otimes 1\big) + \frac{d'}{8}\end{aligned}$$

[14] Recall from §2 that, like most constructions in this paper, both W^f and $\Gamma^{\mathrm{ch}}(\pi, S)$ are the spaces of global sections of underlying sheaves.

For the first two sums above, write $r = t + (r - t)$ and sum over r:

$$\text{1st sum } = -\frac{1}{4}\sum_{t\geq 0} t\,(c\sigma_j)_{-t}(c\sigma_j)_t + \frac{1}{4}\sum_{t\geq 0}(c\sigma_j)_s(c\sigma_k)_t\big(\theta_{jk,-s-t}\otimes 1\big)$$
$$\text{2nd sum} = \frac{1}{4}\sum_{t<0} t\,(c\sigma_j)_t(c\sigma_j)_{-t} - \frac{1}{4}\sum_{t<0}(c\sigma_k)_t(c\sigma_j)_s\big(\theta_{jk,-s-t}\otimes 1\big)$$

For the other two sums above, apply (7.18) and rearrange:

$$\begin{aligned}\text{3rd} + \text{4th sums} &= \frac{1}{2}\bigg(\sum_{r\geq 0>t} - \sum_{r<0\leq t}\bigg) r\, g_{ij,-r+t}g_{ij,r-t}\otimes 1 = \frac{1}{4}\sum_u (u^2-u)\, g_{ij,-u}g_{ij,u}\otimes 1\\ &= -\frac{1}{4}\big(\mathrm{Tr}\,(dg^{-1})_{-1}(dg)\big)_0\otimes 1 = \frac{1}{4}\,\mathrm{Tr}\,(\theta_{-1}\theta)_0\otimes 1\end{aligned}$$

These calculations together yield the following expression for the first term of ℓ'':

$$1\otimes L_0^{C\ell} = -\frac{1}{2}\sum_{t>0} t(c\sigma_j)_{-t}(c\sigma_j)_t + \frac{1}{4}(c\sigma_j)_s(c\sigma_k)_t(\theta_{jk,-s-t}\otimes 1) + \frac{1}{4}\mathrm{Tr}\,(\theta_{-1}\theta)_0\otimes 1 + \frac{d'}{8}\,. \tag{7.19}$$

On the other hand, the second term of ℓ'' can be written as follows by (7.2) and (7.17), where the normal ordering of $e_{i,r-s}e_{j,s}$ guarantees the validity of the subsequent step:

$$\begin{aligned}-\Theta^a_{-r}\otimes t^{C\ell}_{a,r} &= \frac{1}{4}\Theta_{ij,-r}\otimes e_{i,r-s}e_{j,s} = \frac{1}{4}\sum_{s\geq 0}\Theta_{ij,-r}\otimes e_{i,r-s}e_{j,s} - \frac{1}{4}\sum_{s<0}\Theta_{ij,-r}\otimes e_{j,s}e_{i,r-s}\\ &= \frac{1}{4}\bigg(\sum_{s\geq 0}(c\sigma_k)_t(c\sigma_\ell)_u - \sum_{s<0}(c\sigma_\ell)_u(c\sigma_k)_t\bigg)\big((g^{-1}\Theta)_{kj,-s-t}\, g_{j\ell,s-u}\otimes 1\big)\end{aligned}$$

These sums can be handled in a similar way as in the calculation of $1\otimes L_0^{C\ell}$. Let us omit the details and just write down the result:

$$-\Theta^a_{-r}\otimes t^{C\ell}_{a,r} = \frac{1}{4}\,(c\sigma_k)_t(c\sigma_\ell)_u\big((g^{-1}\Theta\, g)_{k\ell,-t-u}\otimes 1\big) + \frac{1}{2}\,\mathrm{Tr}\,\big((g^{-1}\Theta\, g)_{-1}\theta\big)_0\otimes 1. \tag{7.20}$$

Then it follows from (7.19), (7.20) and (6.12) that

$$\ell'' = -\frac{1}{2}\sum_{t>0} t(c\sigma_j)_{-t}(c\sigma_j)_t + \frac{1}{4}(c\sigma_j)_s(c\sigma_k)_t(\Gamma^\sigma_{jk,-s-t}\otimes 1) + \frac{1}{4}\mathrm{Tr}\,(\Gamma^\sigma_{-1}\Gamma^\sigma)_0\otimes 1 + \frac{d'}{8}\,.$$

Combining this with the earlier comment on ℓ' and keeping in mind (7.8)–(7.9), we see that $\ell'+\ell''$ represents the operator (7.16). This proves the claim for L_0^M.

Since the expression in (7.16) is generated by the operators in (7.8)–(7.9), it determines an operator L'_0 on W^f. To show $L'_0 = L^f_0$, we need to verify that L'_0 satisfies (7.14) using only the relations in (7.10), Lemma 7.7 and Propositions 7.8–7.9. This is straightforward (but uninteresting). $\square$

Corollary 7.13. *The linear map $\Phi : W^f \to \Gamma^{\mathrm{ch}}(\pi, S)$ (see §7.10) is an isomorphism.*

Proof. Let $f\in C^\infty(M)$, $\alpha\in\Omega^1(M)$ and $X\in\mathcal{T}(M)$. By definition, Φ is an isomorphism on the lowest weight $d'/8$. Assume that Φ is an isomorphism on all weights up to $(d'/8)+k-1$ for some $k>0$. Let $u\in\Gamma^{\mathrm{ch}}(\pi,S)_k$. For reason of weight as well as (7.10), elements of the form

$$f_n u,\ \ell X_n u,\ cX_n u\ \text{ for } n>0, \qquad \alpha_n u\ \text{ for } n\geq 0 \tag{7.21}$$

belong to the image of Φ; then by Proposition 7.12 so does $ku = (L_0^M - d'/8)u$. This proves the surjectivity of Φ on weight $(d'/8)+k$. On the other hand, let $u\in W^f_k\cap\ker\Phi$. Since the elements (7.21) are also in the kernel of Φ, for reason of weight and (7.10) again they must be trivial; then by Proposition 7.12 so is $ku = (L^f_0 - d'/8)u$. This proves the injectivity of Φ on weight $(d'/8)+k$. By induction, Φ is an isomorphism on all weights. $\square$

The following summarizes our analysis of $\Gamma^{\mathrm{ch}}(\pi, S)$.

Theorem 7.14. *Suppose $M^{2d'}$ is a Riemannian manifold with a spin structure $\pi : P \to M$ and a 3-form H satisfying $dH = \frac{1}{2}\mathrm{Tr}\,(R \wedge R)$, where R is the Riemannian curvature. Let $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P)$ be the associated principal frame $(\widehat{\mathfrak{so}}_{2d'}, \mathrm{Spin}_{2d'})$-algebra (see Theorem 4.14), S the spinor representation of $\widehat{\mathfrak{so}}_{2d'}$, and*

$$\Gamma^{\mathrm{ch}}(\pi, S) = H^{\frac{\infty}{2}+0}\big(\widehat{\mathfrak{so}}_{2d'}\,,\, \mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S\big).$$

Then $\Gamma^{\mathrm{ch}}(\pi, S)$ contains $S(M)$ as a subspace (see §7.4) and admits three particular types of fields $\mathsf{Y}(f, x)$, $\mathsf{Y}(\ell X, x)$, $\mathsf{Y}(cX, x)$ associated to $f \in C^\infty(M)$ and $X \in \mathcal{T}(M)$, whose Fourier modes are represented in $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S$ by the following operators: (see §7.6)

$$f_n \otimes 1, \qquad (\widetilde{X} - \tau_i \widetilde{X}^j \Theta_{ij})_n \otimes 1, \qquad \widetilde{X}^i_{n-r} \otimes e_{i,r}, \qquad n \in \mathbb{Z}.$$

In fact, $\Gamma^{\mathrm{ch}}(\pi, S)$ is generated from $S(M)$ by the actions of these fields, subject only to the relations stated in Propositions 7.8–7.9 (see also (7.10) and Lemma 7.7). Moreover, there is a Virasoro action on $\Gamma^{\mathrm{ch}}(\pi, S)$ of central charge $5d'$, induced by the following operators on $\mathcal{D}^{\mathrm{ch}}_{\Theta,H}(P) \otimes S$: (see §7.11)

$$(\tau_{i,-1}\tau^i)_n \otimes 1 + 1 \otimes L^{C\ell}_n - \Theta^a_{n-r} \otimes t^{C\ell}_{a,r} - \tfrac{1}{4}\mathrm{Tr}\,(\Theta_{-1}\Theta)_n \otimes 1, \qquad n \in \mathbb{Z}. \qquad \square$$

The generators-and-relations description of $\Gamma^{\mathrm{ch}}(\pi, S)$ allows us to define a filtration and identify the associated graded space.

§ 7.15. PBW filtration. Given an increasing sequence of negative integers $\mathfrak{n} = \{n_1 \le \cdots \le n_s < 0\}$, possibly empty, let us write

$$|\mathfrak{n}| = |n_1| + \cdots + |n_s| \quad (0 \text{ if } \mathfrak{n} = \{\}), \qquad \mathfrak{n}(i) = \text{number of times } i \text{ appears in } \mathfrak{n}$$

and regard $\mathfrak{n}$ as a partition of $-|\mathfrak{n}|$. For any nonnegative integer w, let $\mathscr{I}_w$ be the set of triples $(\mathfrak{n}; \mathfrak{m}; \mathfrak{p})$ of such sequences with $|\mathfrak{n}| + |\mathfrak{m}| + |\mathfrak{p}| = w$ and $\mathfrak{m}(i) \le 2d'$ for all $i < 0$. Define a partial ordering on $\mathscr{I}_w$ by declaring that $(\mathfrak{n}; \mathfrak{m}\ \mathfrak{p}) \prec (\mathfrak{n}'; \mathfrak{m}'; \mathfrak{p}')$ if one of the following is true:

- $|\mathfrak{n}| < |\mathfrak{n}'|$, or $|\mathfrak{n}| = |\mathfrak{n}'|$ and $|\mathfrak{m}| < |\mathfrak{m}'|$
- $\mathfrak{n}'$ is a proper subpartition of $\mathfrak{n}$, $\mathfrak{m} = \mathfrak{m}'$ and $\mathfrak{p} = \mathfrak{p}'$
- $\mathfrak{n} = \mathfrak{n}'$, $\mathfrak{m} = \mathfrak{m}'$ and $\mathfrak{p}$ is a proper subpartition of $\mathfrak{p}'$

For example in $\mathscr{I}_3$, a particular ascending chain is

$$(;;-2,-1) \prec (;;-3) \prec (;-1,-1;-1) \prec (-2;-1;) \prec (-1,-1;-1;)$$

while $(;;-1,-1,-1)$ and $(-1,-1,-1;;)$ are the unique minimal and maximal elements.

Given a sequence $\mathfrak{n} = \{n_1 \le \cdots \le n_s < 0\}$ as above and an s-tuple $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$ in $\Omega^1(M)$, or an s-tuple $\boldsymbol{X} = (X_1, \dots, X_s)$ in $\mathcal{T}(M)$, let us introduce the notations

$$\boldsymbol{\alpha}_{\mathfrak{n}} = \alpha_{1,n_1} \cdots \alpha_{s,n_s}, \quad \boldsymbol{cX}_{\mathfrak{n}} = (cX_1)_{n_1} \cdots (cX_s)_{n_s}, \quad \boldsymbol{\ell X}_{\mathfrak{n}} = (\ell X_1)_{n_1} \cdots (\ell X_s)_{n_s} \quad (1 \text{ if } \mathfrak{n} = \{\})$$

which are operators on $\Gamma^{\mathrm{ch}}(\pi, S)$ (see §7.6). It follows from Theorem 7.14 that for $w \ge 0$ we have

$$\Gamma^{\mathrm{ch}}(\pi, S)_w = \mathrm{span}\Big\{\boldsymbol{\ell X}_{\mathfrak{n}} \boldsymbol{cY}_{\mathfrak{m}} \boldsymbol{\alpha}_{\mathfrak{p}}\, s : (\mathfrak{n}; \mathfrak{m}\ \mathfrak{p}) \in \mathscr{I}_w;\ \text{all suitable } \boldsymbol{\alpha}, \boldsymbol{X}, \boldsymbol{Y};\ s \in S(M)\Big\}.$$

Indeed, as $f_n = -\frac{1}{n}(df)_n$ for $f \in C^\infty(M)$ and $n \ne 0$, Propositions 7.8 and 7.9 allow us to express every element of $\Gamma^{\mathrm{ch}}(\pi, S)$ in the indicated form. For $(\mathfrak{n}; \mathfrak{m}; \mathfrak{p}) \in \mathscr{I}_w$ consider the subspaces

$$\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m};\mathfrak{p})} = \mathrm{span}\big\{\boldsymbol{\ell X}_{\mathfrak{n}'} \boldsymbol{cY}_{\mathfrak{m}'} \boldsymbol{\alpha}_{\mathfrak{p}'} s : (\mathfrak{n}'; \mathfrak{m}'; \mathfrak{p}') \preceq (\mathfrak{n}; \mathfrak{m}; \mathfrak{p})\big\} \subset \Gamma^{\mathrm{ch}}(\pi, S)_w$$
$$\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m};\mathfrak{p})} = \mathrm{span}\big\{\boldsymbol{\ell X}_{\mathfrak{n}'} \boldsymbol{cY}_{\mathfrak{m}'} \boldsymbol{\alpha}_{\mathfrak{p}'} s : (\mathfrak{n}'; \mathfrak{m}'; \mathfrak{p}') \prec (\mathfrak{n}; \mathfrak{m}; \mathfrak{p})\big\} \subset \Gamma^{\mathrm{ch}}(\pi, S)_w$$

For the next statement, let O (and O') stand for one of the operators of the form α_n, cX_n or ℓX_n with $n<0$, and fO the corresponding operator $(f\alpha)_n$, $c(fX)_n$ or $\ell(fX)_n$, where $f\in C^\infty(M)$. The subspaces $\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m};\mathfrak{p})}$ and $\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m};\mathfrak{p})}$ have the following properties:

$$\begin{array}{llcl} \text{(i)} & \cdots OO'\cdots\in\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m};\mathfrak{p})} & \Rightarrow & \cdots[O,O']\cdots\in\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m};\mathfrak{p})} \\ \text{(ii)} & \cdots Os\in\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m};\mathfrak{p})} & \Rightarrow & \cdots\big((fO)s-O(fs)\big)\in\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m};\mathfrak{p})}, \qquad s\in S(M) \end{array}$$

Indeed (i) follows from the supercommutation relations in Proposition 7.9 and (ii) from the normal-ordered expansions there as well as Proposition 7.8. Consequently, there is a natural isomorphism

$$\begin{aligned} &\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m};\mathfrak{p})}/\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m};\mathfrak{p})} \\ &\quad\cong\Big(\bigotimes_{i<0}\mathrm{Sym}^{\mathfrak{p}(i)}\Omega^1(M)\Big)\otimes\Big(\bigotimes_{i<0}\mathrm{Sym}^{\mathfrak{n}(i)}\mathcal{T}(M)\Big)\otimes\Big(\bigotimes_{i<0}\wedge^{\mathfrak{m}(i)}\mathcal{T}(M)\Big)\otimes S(M) \end{aligned}$$

where all tensor, symmetric and exterior products are over $C^\infty(M)$. Let q be a formal variable. When all $(\mathfrak{n};\mathfrak{m}\ \mathfrak{p})\in\mathscr{I}_w$ and all $w\geq 0$ are considered, we obtain

$$\begin{aligned} &\bigoplus_{w\geq 0}\Big(q^w\bigoplus_{(\mathfrak{n};\mathfrak{m};\mathfrak{p})\in\mathscr{I}_w}\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m};\mathfrak{p})}/\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m};\mathfrak{p})}\Big) \\ &\quad\cong\Big(\bigotimes_{k\geq 1}\mathrm{Sym}_{q^k}\Omega^1(M)\Big)\otimes\Big(\bigotimes_{k\geq 1}\mathrm{Sym}_{q^k}\mathcal{T}(M)\Big)\otimes\Big(\bigotimes_{k\geq 1}\wedge_{q^k}\mathcal{T}(M)\Big)\otimes S(M) \end{aligned} \tag{7.22}$$

where $\mathrm{Sym}_t=\sum_{n=0}^\infty t^n\mathrm{Sym}^n$ and $\wedge_t=\sum_{n=0}^{2d'}t^n\wedge^n$ as usual. [15]

To conclude this section, let us use an example to illustrate a speculation about $\Gamma^{\mathrm{ch}}(\pi,S)$ that we plan to investigate in a future paper.

§ 7.16. The case of $\mathbb{R}^{2d'}$: (0,1) superconformal structure. When $M=\mathbb{R}^{2d'}$ with the standard spin structure and H is the trivial 3-form, $\Gamma^{\mathrm{ch}}(\pi,S)$ can be identified with $\mathcal{D}^{\mathrm{ch}}(\mathbb{R}^{2d'})\otimes S$. Recall the definition of $\mathcal{D}^{\mathrm{ch}}(\mathbb{R}^{2d'})$ from §2.2. Then it is easy to check that the following operators on $\mathcal{D}^{\mathrm{ch}}(\mathbb{R}^{2d'})\otimes S$

$$\begin{aligned} L_n&=\tfrac14\big((\partial_i+db^i)_{-1}(\partial_i+db^i)\big)_n\otimes 1 \\ \bar L_n&=-\tfrac14\big((\partial_i-db^i)_{-1}(\partial_i-db^i)\big)_n\otimes 1+1\otimes L_n^{C\ell}, \qquad n\in\mathbb{Z} \\ \bar G_n&=\tfrac12(\partial_i-db^i)_{n-r}\otimes e_{i,r} \end{aligned}$$

satisfy the $(0,1)$ (Ramond) superconformal algebra of central charges $(2d',3d')$, namely

$$\begin{aligned} &[L_n,L_m]=(n-m)L_{n+m}+\tfrac{d'}{6}(n^3-n)\delta_{n,-m} \\ &[L_n,\bar L_m]=[L_n,\bar G_m]=0 \\ &[\bar L_n,\bar L_m]=(n-m)\bar L_{n+m}+\tfrac{d'}{4}(n^3-n)\delta_{n,-m} \\ &[\bar L_n,\bar G_m]=\big(\tfrac{n}{2}-m\big)\bar G_{n+m}, \qquad [\bar G_n,\bar G_m]=2\bar L_{n+m}+\tfrac{d'}{4}(4n^2-1)\delta_{n,-m} \end{aligned}$$

Notice that the diagonal Virasoro action given by $L_n+\bar L_n$ is precisely the one described in §7.11.

Let us consider the operator $\not{D}=2\bar G_0$ and its kernel. Since $\not{D}^2=4(\bar L_0-d'/8)$, on $\ker\not{D}$ we have $\bar L_0=d'/8$. Notice that $\mathcal{D}^{\mathrm{ch}}(\mathbb{R}^{2d'})\otimes S$ can be generated from $S(\mathbb{R}^{2d'})=C^\infty(\mathbb{R}^{2d'})\otimes S_0$ by the actions of $(\partial_i+db^i)_n$, $(\partial_i-db^i)_n$ and $e_{i,n}$ for $n<0$. Also notice that $\bar L_0\geq d'/8$ on $S(\mathbb{R}^{2d'})$ and

$$[\bar L_0,(\partial_i+db^i)_n]=0, \qquad [\bar L_0,(\partial_i-db^i)_n]=-n(\partial_i+db^i)_n, \qquad [\bar L_0,e_{i,n}]=-ne_{i,n}.$$

[15] If the partial ordering on $\mathscr{I}_w$ is extended to a total ordering, the latter will induce a filtration on $\Gamma^{\mathrm{ch}}(\pi,S)_w$ whose associated graded space is the coefficient of q^w in (7.22).

All these imply that $\ker \bar{D}$ must be contained in the subspace generated from $S(\mathbb{R}^{2d'})$ by $(\partial_i + db^i)_n$ only. This subspace can be identified with

$$\left(\bigotimes_{k\geq 1} \mathrm{Sym}_{q^k}\mathcal{T}(\mathbb{R}^{2d'})\right) \otimes S(\mathbb{R}^{2d'}) \tag{7.23}$$

where the various products are taken over $C^\infty(\mathbb{R}^{2d'})$ and the power of q corresponds to the eigenvalue of $L_0 + \bar{L}_0 - d'/8$. Since $\bar{D}$ also commutes with $(\partial_i + db^i)_n$, it follows that the restriction of $\bar{D}$ to (7.23) is simply a classical twisted Dirac operator on $\mathbb{R}^{2d'}$. Moreover, all L_n commute with $\bar{D}$ and hence induce a Virasoro action on $\ker \bar{D}$ of central charge $2d'$.

Remark. The author speculates that, in general, the Virasoro action on $\Gamma^{\mathrm{ch}}(\pi, S)$ described in §7.11 always extends to an action of the $(0,1)$ superconformal algebra (which includes a Dirac-Ramond operator $\bar{D} = 2\bar{G}_0$); and if M is closed, a more sophisticated version of the argument above would prove that

$$\begin{aligned}
\text{Virasoro character of } \ker \bar{D} &:= q^{-d'/12} \times \text{supertrace of } q^{L_0} \text{ on } \ker \bar{D} \\
&= q^{-d'/12} \times \text{supertrace of } q^{L_0+\bar{L}_0-d'/8} \text{ on } \ker \bar{D} \\
&= q^{-d'/12} \times \hat{A}\left(M, \bigotimes_{k\geq 1} \mathrm{Sym}_{q^k} TM \otimes \mathbb{C}\right) \\
&= \eta(q)^{-2d'} \times \text{Witten genus of } M
\end{aligned}$$

where $\eta(q)$ the Dedekind η-function. This would provide at least a partial mathematical formulation of the original, physical interpretation of the Witten genus. [Wit87, Wit88] It is believed that such a description of the Witten genus may lead to various geometric applications as well as a better understanding of its family version in elliptic cohomology. [Hop02]

Appendix §A. Vertex Algebroids

Given a vertex algebra (with nonnegative integer-valued weights), its data involving only the two lowest weights can be repackaged into what is called a vertex algebroid. Admittedly, the definition of a vertex algebroid is rather complicated, but it serves as a convenient tool for dealing with the vertex algebras in this paper. This appendix reviews, mostly without proof, the category of vertex algebroids and the adjoint functors between vertex algebras and vertex algebroids. For more details, the reader is referred to the original work [GMS04].

Definition A.1. An **extended Lie algebroid** $(A, \Omega, \mathcal{T})$ consists of:

- a commutative, associative $\mathbb{C}$-algebra with unit $(A, \mathbf{1})$
- an A-module Ω, together with an A-derivation $d : A \to \Omega$ such that $\Omega = A \cdot dA$
- another A-module $\mathcal{T}$, equipped with a Lie bracket $[\ ]$
- an A-linear map of Lie algebras $\mathcal{T} \to \operatorname{End} A$, denoted by $X \mapsto X$
- a $\mathbb{C}$-linear map of Lie algebras $\mathcal{T} \to \operatorname{End} \Omega$, denoted by $X \mapsto L_X$
- an A-bilinear pairing $\Omega \times \mathcal{T} \to A$, denoted by $(\alpha, X) \mapsto \alpha(X)$

Furthermore, it is required that:

- the $\mathcal{T}$-actions on A and Ω commute with d
- the $\mathcal{T}$-actions on A, Ω and $\mathcal{T}$ (via $[\ ]$) satisfy the Leibniz rule with respect to A-multiplication
- $df(X) = Xf$ for $f \in A$ and $X \in \mathcal{T}$

Definition A.2. A **map of extended Lie algebroids** $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ is simply a map of ordered triples that respects the extended Lie algebroid structures. Composition of maps is defined in the obvious way.

Definition A.3. A **vertex algebroid** $(A, \Omega, \mathcal{T}, \bullet, \{\ \}, \{\ \}_\Omega)$ consists of an extended Lie algebroid $(A, \Omega, \mathcal{T})$ together with three $\mathbb{C}$-bilinear maps

$$\bullet : \mathcal{T} \times A \to \Omega, \qquad \{\ \} : \mathcal{T} \times \mathcal{T} \to A, \qquad \{\ \}_\Omega : \mathcal{T} \times \mathcal{T} \to \Omega$$

that satisfy the following identities:

- $\{X, Y\} = \{Y, X\}$
- $d\{X, Y\} = \{X, Y\}_\Omega + \{Y, X\}_\Omega$
- $X \bullet (fg) - (gX) \bullet f - f(X \bullet g) = -(Xf)dg$
- $\{X, fY\} - f\{X, Y\} = -(Y \bullet f)(X) + [X, Y]f$
- $\{X, fY\}_\Omega - f\{X, Y\}_\Omega = -L_X(Y \bullet f) + [X, Y] \bullet f + Y \bullet (Xf)$
- $X\{Y, Z\} - \{[X, Y], Z\} - \{Y, [X, Z]\} = \{X, Y\}_\Omega(Z) + \{X, Z\}_\Omega(Y)$
- $L_X\{Y, Z\}_\Omega - L_Y\{X, Z\}_\Omega + L_Z\{X, Y\}_\Omega + \{X, [Y, Z]\}_\Omega - \{Y, [X, Z]\}_\Omega - \{[X, Y], Z\}_\Omega$
  $= d\Big(\{X, Y\}_\Omega(Z)\Big)$

for $f, g \in A$ and $X, Y, Z \in \mathcal{T}$.

Remark. This definition is equivalent to, but slightly different from both the original one in [GMS04] with the notations $(\gamma, \langle\ \rangle, c)$, and the one in [Che11] with the notations $(*, \{\ \}, \{\ \}_\Omega)$. The various notations are related as follows:

$$X \bullet f = -\gamma(f, X) + dXf = f * X + dXf$$
$$\{X, Y\} = \langle X, Y\rangle, \qquad \{X, Y\}_\Omega = -c(X, Y) + \frac{1}{2}\langle X, Y\rangle.$$

In this paper we adopt the above definition to simplify the description of certain vertex algebras.

Definition A.4. A **map of vertex algebroids**

$$(\varphi, \Delta) : (A, \Omega, \mathcal{T}, \bullet, \{\ \}, \{\ \}_\Omega) \to (A', \Omega', \mathcal{T}', \bullet', \{\ \}', \{\ \}'_\Omega)$$

consists of a map of extended Lie algebroids $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ together with a $\mathbb{C}$-linear map $\Delta : \mathcal{T} \to \Omega'$ such that

- $\varphi X \bullet' \varphi f - \varphi(X \bullet f) = \Delta(fX) - (\varphi f)\Delta(X)$
- $\{\varphi X, \varphi Y\}' - \varphi\{X, Y\} = -\Delta(X)(\varphi Y) - \Delta(Y)(\varphi X)$
- $\{\varphi X, \varphi Y\}'_\Omega - \varphi\{X, Y\}_\Omega = -L_{\varphi X}\Delta(Y) + L_{\varphi Y}\Delta(X) - d\big(\Delta(X)(\varphi Y)\big) + \Delta([X, Y])$

for $f \in A$ and $X, Y \in \mathcal{T}$. Composition of maps is defined by

$$(\varphi', \Delta') \circ (\varphi, \Delta) = (\varphi'\varphi,\ \varphi'\Delta + \Delta'\varphi|_{\mathcal{T}}).$$

§ A.5. From vertex algebras to vertex algebroids: objects. [16] Given a vertex algebra $(V, \mathbf{1}, T, Y)$, consider the following subquotient spaces

$$A := V_0, \qquad \Omega := A_0(TA), \qquad \mathcal{T} := V_1/\Omega.$$

Choose a splitting $s : \mathcal{T} \to V_1$ of the quotient map to obtain an identification of vector spaces

$$\Omega \oplus \mathcal{T} \cong V_1, \qquad (\alpha, X) \mapsto \alpha + s(X). \tag{A.1}$$

The part of vertex algebra structure on V involving only the two lowest weights consists of an element $\mathbf{1} \in V_0$, a linear map $T : V_0 \to V_1$, and eight bilinear maps

$$V_i \times V_j \to V_k, \quad (u, v) \mapsto u_{j-k}v, \qquad \text{for } i, j, k = 0, 1$$

satisfying a set of (Borcherds) identities. All these data, when rephrased in terms of the identification (A.1), are equivalent to a vertex algebroid $(A, \Omega, \mathcal{T}, \bullet, \{\ \}, \{\ \}_\Omega)$. The extended Lie algebroid $(A, \Omega, \mathcal{T})$ consists of precisely those data that are independent of the choice of s, namely

$$\begin{array}{lll} fg := f_0 g & f\alpha := f_0\alpha & fX := f_0 s(X) \bmod \Omega \\ Xf := s(X)_0 f & L_X\alpha := s(X)_0\alpha & [X, Y] := s(X)_0 s(Y) \bmod \Omega \\ df := Tf & \alpha(X) := \alpha_1 s(X) & \end{array}$$

[17] for $f, g \in A$, $\alpha \in \Omega$ and $X, Y \in \mathcal{T}$. The rest of the vertex algebroid structure is given by

$$\begin{aligned} X \bullet f &:= s(X)_{-1}f - s(fX) \\ \{X, Y\} &:= s(X)_1 s(Y) \\ \{X, Y\}_\Omega &:= s(X)_0 s(Y) - s([X, Y]) \end{aligned} \tag{A.2}$$

for $f \in A$ and $X, Y \in \mathcal{T}$.

§ A.6. From vertex algebras to vertex algebroids: morphisms. Let $\Phi : V \to V'$ be a map of vertex algebras. Then let $(A, \Omega, \mathcal{T}, \cdots)$ be the vertex algebroid associated to V and a splitting $s : \mathcal{T} \to V_1$; and similarly $(A', \Omega', \mathcal{T}', \cdots)$ associated to V' and $s' : \mathcal{T}' \to V'_1$. The part of data of Φ involving only the two lowest weights, when rephrased in terms of identifications like (A.1), are equivalent to a map of vertex algebroids (φ, Δ). It consists of the obvious map of ordered triples $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ induced by Φ, and a map $\Delta : \mathcal{T} \to \Omega'$ defined by

$$\Delta(X) = \Phi s(X) - s'(\varphi X).$$

[16] The reader is encouraged to first take a quick look of §1.2 about our conventions for vertex algebras.
[17] For example, the definition of Xf is indeed independent of s because $\alpha_0 f = 0$ for $f \in A$ and $\alpha \in \Omega$.

§ A.7. From vertex algebroids to vertex algebras: objects. Let $(A, \Omega, \mathcal{T}, \bullet, \{\ \}, \{\ \}_\Omega)$ be a vertex algebroid. In this discussion, every statement concerns or applies to all $f, g \in A$; $\alpha, \beta \in \Omega$; and $X, Y \in \mathcal{T}$. Define a unital associative algebra $\mathcal{U}$ with generators f_n, α_n, X_n, $n \in \mathbb{Z}$, subject to the relations

$$\begin{array}{l} f \mapsto f_n,\ \alpha \mapsto \alpha_n,\ X \mapsto X_n \ \text{ are linear} \\ \mathbf{1}_n = \delta_{n,0}, \qquad (df)_n = -nf_n, \qquad [f_n, g_m] = [f_n, \alpha_m] = [\alpha_n, \beta_m] = 0 \\ [X_n, f_m] = (Xf)_{n+m}, \qquad [X_n, \alpha_m] = (L_X\alpha)_{n+m} + n\alpha(X)_{n+m} \\ [X_n, Y_m] = [X, Y]_{n+m} + (\{X, Y\}_\Omega)_{n+m} + n\{X, Y\}_{n+m} \end{array} \tag{A.3}$$

for $n, m \in \mathbb{Z}$. The subalgebra $\mathcal{U}_+ \subset \mathcal{U}$ generated by $\{f_n\}_{n>0}$ and $\{\alpha_n, X_n\}_{n\geq 0}$ has a trivial action on $\mathbb{C}$. Let $\widetilde{V} := \mathcal{U} \otimes_{\mathcal{U}_+} \mathbb{C}$ be the induced $\mathcal{U}$-module and $V := \widetilde{V}/\sim$ the quotient module obtained by imposing the following relations for $v \in \widetilde{V}$:

$$\begin{array}{lcl} (fg)_n v & \sim & \sum_{k\in\mathbb{Z}} f_{n-k} g_k v \\ (f\alpha)_n v & \sim & \sum_{k\in\mathbb{Z}} f_{n-k} \alpha_k v \\ (fX)_n v & \sim & \sum_{k\geq 0} f_{n-k} X_k v + \sum_{k<0} X_k f_{n-k} v - (X \bullet f)_n v \end{array} \tag{A.4}$$

Notice that the summations are always finite. It follows from the axioms of a vertex algebroid that (A.3)–(A.4) are consistent. [18] Define a vertex algebra structure on V as follows. The vacuum $\mathbf{1} \in V$ is the coset of $1 \otimes 1 \in \widetilde{V}$. The infinitesimal translation T and weight operator L_0 are determined by

$$\begin{array}{llll} T\mathbf{1} = 0, & [T, f_n] = (1-n)f_{n-1}, & [T, \alpha_n] = -n\alpha_{n-1}, & [T, X_n] = -nX_{n-1} \\ L_0\mathbf{1} = 0, & [L_0, f_n] = -nf_n, & [L_0, \alpha_n] = -n\alpha_n, & [L_0, X_n] = -nX_n \end{array}$$

which are consistent with (A.3)–(A.4); notice that actions of f_n, α_n, X_n change weights by $-n$. Identify $A, \Omega, \mathcal{T}$ as subspaces of V via $f = f_0\mathbf{1}$, $\alpha = \alpha_{-1}\mathbf{1}$, $X = X_{-1}\mathbf{1}$, and associate to them the fields

$$\textstyle\sum_n f_n z^{-n}, \qquad \sum_n \alpha_n z^{-n-1}, \qquad \sum_n X_n z^{-n-1}$$

which are mutually local by (A.3). Now apply the Reconstruction Theorem [FB04].

Remark. Suppose V' is a vertex algebra whose associated vertex algebroid is $(A, \Omega, \mathcal{T}, \bullet, \{\ \}, \{\ \}_\Omega)$. By construction, there is a canonical map of vertex algebras $V \to V'$. If it is surjective (resp. bijective), then V' is said to be generated (resp. freely generated) by the vertex algebroid.

§ A.8. From vertex algebroids to vertex algebras: morphisms. A map of vertex algebroids

$$(\varphi, \Delta) : (A, \Omega, \mathcal{T}, \cdots) \to (A', \Omega', \mathcal{T}', \cdots)$$

induces a map $\Phi : V \to V'$ between the freely generated vertex algebras by

$$\begin{array}{lll} \Phi f = \varphi f, & \Phi\alpha = \varphi\alpha, & \Phi X = \varphi X + \Delta(X) \\ \Phi \circ f_n = (\Phi f)_n \circ \Phi, & \Phi \circ \alpha_n = (\Phi\alpha)_n \circ \Phi, & \Phi \circ X_n = (\Phi X)_n \circ \Phi \end{array}$$

for $f \in A$, $\alpha \in \Omega$, $X \in \mathcal{T}, n \in \mathbb{Z}$. Indeed, these equations are consistent with (A.3)–(A.4).

Lemma A.9. *Given a vertex algebroid $(A, \Omega, \mathcal{T}, \bullet, \{\ \}, \{\ \}_\Omega)$, an isomorphism of extended Lie algebroids $\varphi : (A, \Omega, \mathcal{T}) \to (A', \Omega', \mathcal{T}')$ and a $\mathbb{C}$-linear map $\Delta : \mathcal{T} \to \Omega'$, if we define*

$$\bullet' : A' \times \mathcal{T}' \to \Omega', \qquad \{\ \}' : \mathcal{T}' \times \mathcal{T}' \to A', \qquad \{\ \}'_\Omega : \mathcal{T}' \times \mathcal{T}' \to \Omega'$$

by the equations in Definition A.4, then $(A', \Omega', \mathcal{T}', \bullet', \{\ \}', \{\ \}'_\Omega)$ is a vertex algebroid and (φ, Δ) is by construction an isomorphism between the two vertex algebroids. □

[18] For example, $[X_n, (fY)_m]$ can be evaluated by either taking the commutator first or expanding $(fY)_m$ first. The resulting identity is already implied by the vertex algebroid axioms and does not lead to a new relation.

Example A.10. The vertex algebroids associated to a Lie algebra. Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a vertex algebroid of the form $(\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0)$ with $\mathfrak{g}$ acting trivially on $\mathbb{C}$. The second, fourth and last components are trivial by necessity. By Definition A.3, the conditions on $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ are

$$\lambda(X, Y) = \lambda(Y, X), \qquad \lambda([X, Y], Z) + \lambda(Y, [X, Z]) = 0$$

i.e. it is an invariant symmetric bilinear form on $\mathfrak{g}$. Let

$$V_\lambda(\mathfrak{g}) = \text{vertex algebra freely generated by } (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0).$$

In the case $\mathfrak{g}$ is finite-dimensional, simple and λ equals k times the normalized Killing form, this is the same as the affine vertex algebra $V_k(\mathfrak{g})$. [Kac98, FB04]

§ A.11. PBW filtration of a freely generated vertex algebra. Given an increasing sequence of negative integers $\mathfrak{n} = \{n_1 \leq \cdots \leq n_s < 0\}$, possibly empty, we write

$$|\mathfrak{n}| = |n_1| + \cdots + |n_s| \quad (0 \text{ if } \mathfrak{n} = \{\}), \qquad \mathfrak{n}(i) = \text{number of times } i \text{ appears in } \mathfrak{n}$$

and regard $\mathfrak{n}$ as a partition of $-|\mathfrak{n}|$. For $w \geq 0$, let $\mathscr{I}_w$ be the set of pairs $(\mathfrak{n}; \mathfrak{m})$ of such sequences that satisfy $|\mathfrak{n}| + |\mathfrak{m}| = w$. Define a partial ordering on $\mathscr{I}_w$ by declaring that $(\mathfrak{n}; \mathfrak{m}) \prec (\mathfrak{n}'; \mathfrak{m}')$ if

- $|\mathfrak{n}| < |\mathfrak{n}'|$, or
- $\mathfrak{n}'$ is a proper subpartition of $\mathfrak{n}$ and $\mathfrak{m} = \mathfrak{m}'$, or
- $\mathfrak{n} = \mathfrak{n}'$ and $\mathfrak{m}$ is a proper subpartition of $\mathfrak{m}'$

For example, in $\mathscr{I}_3$ a particular chain is given by $(; -2, -1) \prec (; -3) \prec (-2; -1) \prec (-1, -1; -1)$, while $(; -1, -1, -1)$ and $(-1, -1, -1;)$ are the unique minimal and maximal elements.

Consider the vertex algebra V constructed in §A.7. Given a sequence $\mathfrak{n} = \{n_1 \leq \cdots \leq n_s < 0\}$ as above and an s-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ in Ω, or an s-tuple $\boldsymbol{X} = (X_1, \ldots, X_s)$ in $\mathcal{T}$, let

$$\boldsymbol{\alpha}_\mathfrak{n} = \alpha_{1,n_1} \cdots \alpha_{s,n_s}, \qquad \boldsymbol{X}_\mathfrak{n} = X_{1,n_1} \cdots X_{s,n_s} \qquad (1 \text{ if } \mathfrak{n} = \{\})$$

as operators on V. It follows from the relations in (A.3) that

$$V_w = \operatorname{span}\Big\{ \boldsymbol{X}_\mathfrak{n} \boldsymbol{\alpha}_\mathfrak{m} f : (\mathfrak{n}; \mathfrak{m}) \in \mathscr{I}_w; \text{ all suitable } \boldsymbol{\alpha}, \boldsymbol{X};\ f \in A \Big\}.$$

For each $(\mathfrak{n}, \mathfrak{m}) \in \mathscr{I}_w$ define the subspaces

$$\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})} = \operatorname{span}\big\{ \boldsymbol{X}_{\mathfrak{n}'} \boldsymbol{\alpha}_{\mathfrak{m}'} f : (\mathfrak{n}'; \mathfrak{m}') \preceq (\mathfrak{n}; \mathfrak{m}) \big\} \subset V_w$$
$$\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})} = \operatorname{span}\big\{ \boldsymbol{X}_{\mathfrak{n}'} \boldsymbol{\alpha}_{\mathfrak{m}'} f : (\mathfrak{n}'; \mathfrak{m}') \prec (\mathfrak{n}; \mathfrak{m}) \big\} \subset V_w$$

For the next statement, let O (and O') stand for an operator of the form α_n or X_n with $n < 0$, and fO the corresponding operator $(f\alpha)_n$ or $(fX)_n$, where $f \in A$. Observe that the subspaces just defined have the following properties:

$$\begin{array}{llcll} \text{(i)} & \cdots OO' \cdots \in \mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})} & \Rightarrow & \cdots [O, O'] \cdots \in \mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})} & \\ \text{(ii)} & \cdots Og \in \mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})} & \Rightarrow & \cdots \big((fO)g - O(fg)\big) \in \mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})}, & f, g \in A \end{array}$$

Indeed, (i) follows from (A.3) and (ii) from (A.4). These properties imply that

$$\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})} / \mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})} \cong \left(\bigotimes_{i<0} \operatorname{Sym}^{\mathfrak{n}(i)} \mathcal{T} \right) \otimes \left(\bigotimes_{i<0} \operatorname{Sym}^{\mathfrak{m}(i)} \Omega \right)$$

where the tensor and symmetric products are over A. Let q be a formal variable. When all $(\mathfrak{n};\mathfrak{m}) \in \mathscr{I}_w$ and all $w \geq 0$ are considered, we obtain an isomorphism

$$\bigoplus_{w\geq 0}\left(q^w \bigoplus_{(\mathfrak{n};\mathfrak{m})\in\mathscr{I}_w} \mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})}/\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})}\right) \cong \left(\bigotimes_{k\geq 1}\mathrm{Sym}_{q^k}\mathcal{T}\right)\otimes\left(\bigotimes_{k\geq 1}\mathrm{Sym}_{q^k}\Omega\right)$$

where $\mathrm{Sym}_t = \sum_{n=0}^{\infty} t^n \mathrm{Sym}^n$ as usual. [19] The subspaces $\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})}$, $\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})}$ and the above isomorphisms are natural, i.e. respected by maps described in §A.8.

Remark. Let V' be a vertex algebra. Suppose V as considered above is freely generated by the vertex algebroid associated to V', and $\Phi : V \to V'$ is the canonical map. Then we will also use the notations $\mathcal{F}_{\preceq(\mathfrak{n};\mathfrak{m})}$ and $\mathcal{F}_{\prec(\mathfrak{n};\mathfrak{m})}$ for their images under Φ.

Lemma A.12. *If a vertex algebra V has a conformal element $\nu \in \mathcal{F}_{\preceq(-1;-1)} \subset V_2$, then it is generated by its associated vertex algebroid.*

Proof. Consider the vertex algebroid $(A, \Omega, \mathcal{T}, \cdots)$ associated to V and the vertex subalgebra $V' \subset V$ it generates. Let $f, g \in A$; $\alpha, \beta \in \Omega$; and $X \in \mathcal{T}$. By definition, $V'_0 = V_0$ and $V'_1 = V_1$. Suppose $V'_i = V_i$ for $i \leq k-1$, for some positive k, and let $u \in V_k$. It suffices to prove that $u \in V'$, so that $V' = V$.

Clearly $\alpha_r u$, $X_r u \in V_{k-r} \subset V'$ for $r > 0$. Also, it follows from

$$(f dg)_0 = \sum_{s\in\mathbb{Z}} f_{-s}(dg)_s = -\sum_{s>0} s f_{-s} g_s + \sum_{s>0} s g_{-s} f_s \tag{A.5}$$

that we have $(f dg)_0 u \in V'$. Hence in fact

$$\alpha_r u \in V' \text{ for } r \geq 0, \qquad X_r u \in V' \text{ for } r > 0.$$

This easily implies that $(\alpha_{-1}\beta)_0 u$, $(\alpha_{-2}\mathbf{1})_0 u$ and $(X_{-1}\alpha)_0 u$ must all belong to V'. By assumption, ν is a sum of elements of the form $\alpha_{-1}\beta$, $\alpha_{-2}\mathbf{1}$ and $X_{-1}\alpha$, so that $ku = L_0 u = \nu_0 u \in V'$. Since $k > 0$, we have $u \in V'$ as desired. ☐

Lemma A.13. *If a vertex algebra V has a conformal element $\nu \in \mathcal{F}_{\preceq(-1;-1)} \subset V_2$, then it has no nontrivial ideal consisting only of positive weights.*

Proof. Use the same notations as in the proof of Lemma A.12. Let $I \subset V$ an ideal with $I_0 = 0$. Suppose $I_i = 0$ for $i \leq k-1$, for some positive k, and let $u \in I_k$. It suffices to prove that $u = 0$.

Clearly $\alpha_r u$, $X_r u \in I_{k-r} = 0$ for $r > 0$. Also, it follows from (A.5) that $(f dg)_0 u = 0$. Hence in fact

$$\alpha_r u = 0 \text{ for } r \geq 0, \qquad X_r u = 0 \text{ for } r > 0.$$

This easily implies that $(\alpha_{-1}\beta)_0 u = (\alpha_{-2}\mathbf{1})_0 u = (X_{-1}\alpha)_0 u = 0$. By assumption, ν is a sum of elements of the form $\alpha_{-1}\beta$, $\alpha_{-2}\mathbf{1}$ and $X_{-1}\alpha$, so that $ku = L_0 u = \nu_0 u = 0$. Since $k > 0$, we have $u = 0$ as desired. ☐

[19] If we extend the partial ordering on $\mathscr{I}_w$ to a total ordering, the latter will induce in an obvious way a filtration on V_w whose associated graded space is the coefficient of q^w.

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School of Mathematics & Statistics, University of Sheffield, Hicks Building, Sheffield S3 7RH, U.K.
Email address: p.cheung@sheffield.ac.uk